Zeta function of a graph revisited

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Abstract

This talk reviews some new combinatorial and algebraic aspects of the zeta function of a finite graph.

1 Introduction

The \( \zeta_G(z) \) function (also called the Ihara zeta function) of a finite connected and oriented graph \( G \) (the graph may have multiple edges and loops) is formally defined by

\[
\zeta_G(z) = \prod_{\lbrack p \rbrack} (1 - z^{N(p)})^{-1}
\]  

where the product is over the equivalence classes of non periodic backtrack-less and tail-less closed paths (cycles, for short) in \( G \), \( N(p) \) the length of a cycle in \( \lbrack p \rbrack \). It can be rewritten as

\[
\zeta_G(z)^{-1} = \prod_{N=1}^{+\infty} (1 - z^N)^{\Omega(N,T)} = \det(I - zT),
\]

\[
\Omega(N,T) = \frac{1}{N} \sum_{g \mid N} \mu(g) \text{Tr} T^N g.
\]

where \( N \) is a positive integer, \( \mu \) the Möbius function: a) \( \mu(+1) = +1 \), b) \( \mu(g) = 0 \), if \( g = p_1^{e_1} \ldots p_q^{e_q} \), \( p_1, \ldots, p_q \) primes, and any \( e_i > 1 \), c) \( \mu(p_1 \ldots p_q) = (-1)^q \). \( \Omega(N,T) \) is the number of equivalence classes of cycles of length \( N \) in \( G \), \( T \) is the edge adjacency matrix of \( G \), \( g \) ranges over the positive divisors of \( N \). A great deal of work has been done on this function. I refer the reader to [7], [8] and [9] for a comprehensive overview and references therein.

There is a remarkable resemblance between relations (1.2) and (1.3) with the famous \textit{Witt identity} and the \textit{Witt formula}, respectively, which are the following:

\[
\prod_{N=1}^{+\infty} (1 - z^N)^{M(N,R)} = 1 - Rz,
\]

\[
M(N,R) = \frac{1}{N} \sum_{g \mid N} \mu(g) R^N g.
\]
where $R$ is a real number. See the introduction of [4] for a nice account about the Witt relations. The Witt formula is also called the *necklace polynomial* because it solves the problem of counting inequivalent nonperiodic colorings of a necklace with $N$ beads with at most $R$ colors. See [10]. Consider the graph with a single vertex and $R$ loops hooked to it and with the edges directed, say, counter clockwise. Witt formula gives the number of classes of equivalence of cycles of length $N$ that traverse the graph counter clockwise [6]. Another interpretation of the Witt formula is as a dimension formula for the homogenous subspaces of a graded free Lie algebra. The Poincaré-Witt-Birkoff theorem then implies the Witt identity. See [5]. In [10] Metropolis and Rota proved that the Witt polynomials satisfy several identities which they used to prove several important results about the *Necklace Algebra*, *Witt vectors*, etc. Natural questions to ask are: Relations (1.2) and (1.3) can be related to a coloring problem? Is there a connection with free Lie algebras? Do the $\Omega$’s satisfy some identities? It turns out that all these questions have positive answers as I have shown in [1]. These are new aspects of the zeta function of a graph some of which are reviewed in this talk.

In section 2.1 I show that the $\Omega$’s satisfy several identities analogous to those satisfied by the Witt polynomials. In section 2.2 $\Omega$ and the coefficients of the polynomial in $z$ given by $\det(1 - zT)$ are interpreted as the dimensions of certain vector spaces associated to a free Lie superalgebra. See [1] for the details. The results were obtained using ideas from [2], [3], [7] and [10].

## 2 Results

### 2.1 Some identities

In [10] Metropolis and Rota proved that the Witt polynomials satisfy several important identities which they used to build the necklace algebra. Theorem 2.1 shows that $\Omega$ satisfies similar identities. Theorem 2.2 gives a generalization of the classical *Strehl identity* [11].

**Theorem 2.1.** Given the matrices $T_1$ and $T_2$ denote by $T_1 \otimes T_2$ the Kronecker product of $T_1$ and $T_2$. Then,

$$
\sum_{[s,t]=N} (s,t)\Omega(s,T_1)\Omega(t,T_2) = \Omega(N,T_1 \otimes T_2), \quad (2.1)
$$

$(s,t)$ is the maximum common divisor of $s$ and $t$. The summation is over the set of all positive integers $s,t$ such that $[s,t]=N$, $[s,t]$ the least common multiple of $s,t$. Also,

$$
\Omega(N,T^l) = \sum_{[l,t]=N^l} \frac{t}{N^l} \Omega(t,T). \quad (2.2)
$$

and

$$
(r,s)\Omega(N, T_1^{s/(r,s)} \otimes T_2^{r/(r,s)}) = \sum (rp,sq)\Omega(p,T_1)\Omega(q,T_2) \quad (2.3)
$$

The sum is over $p,q$ such that $pq/(pr,qs) = N/(r,s)$. 


Theorem 2.2.

\[
\prod_{k \geq 1} \left[ \frac{1}{\det(1 - z^k T_1)} \right]^{\Omega(k,T_2)} = \prod_{j \geq 1} \left[ \frac{1}{\det(1 - z^j T_2)} \right]^{\Omega(j,T_1)}
\]  \hspace{1cm} (2.4)

It follows from the latter result that

\[
\prod_{k \geq 1} \left[ \zeta_{G_1}(z^k) \right]^{\Omega(k,T_2)} = \prod_{j \geq 1} \left[ \zeta_{G_2}(z^j) \right]^{\Omega(j,T_1)}
\]  \hspace{1cm} (2.5)

Theorem 2.3.

\[
\prod_{n \geq 1} \frac{1}{1 - \Omega(n,T) z^n} = \left( \frac{1}{1 - z} \right)^\alpha
\]  \hspace{1cm} (2.6)

\[
\alpha = \sum_{d \mid n} d \left[ \Omega(d,T) \right]^{n/d}
\]  \hspace{1cm} (2.7)

### 2.2 Dimension formulas

Theorem (2.4) below together with results from [3] imply that (1.2) and (1.3) can naturally be interpreted as data associated to a Lie superalgebra.

**Theorem 2.4.** Define \( g(z) := \sum_{N=1}^{\infty} \frac{\Tr T^N}{N} z^N \). Then,

\[
\prod_{N=1}^{\infty} (1 - z^N)^\pm \Omega(N,T) = e^{\mp g(z)} = [\det(1 - zT)]^{\mp N} = 1 + \sum_{i=1}^{\infty} c_\pm(i) z^i,
\]  \hspace{1cm} (2.8)

where

\[
c_\pm(i) = \sum_{m=1}^{i} \lambda_\pm(m) \sum_{a_1 + 2a_2 + \ldots + ia_i = i} \prod_{k=1}^{i} \frac{(\Tr T^k)^{a_k}}{a_k! k^{a_k}}
\]  \hspace{1cm} (2.9)

with \( \lambda_+(m) = (-1)^{m+1}, \lambda_-(m) = +1, c_+(i) = 0 \) for \( i > 2|E| \), and \( c_-(i) \geq 0 \). Furthermore,

\[
\Tr T^N = N \sum_{s \in S(N)} (\pm 1)^{|s|+1} \frac{|s|!}{s!} \prod_{i=1}^{s} c_\pm(i)^{s_i}
\]  \hspace{1cm} (2.10)

where \( S(N) = \{ s = (s_i)_{i \geq 1} \mid s_i \in \mathbb{Z}_{\geq 0}, \sum s_i = N \} \) and \( |s| = \sum s_i, \ s! = \prod s_i! \).

In section 2.3 of [3], given a formal power series \( \sum_{i=1}^{\infty} t_i z^i \) with \( t_i \in \mathbb{Z} \), for all \( i \geq 1 \), the coefficients in the series are interpreted as the superdimensions of a \( \mathbb{Z}_{\geq 0} \)-graded superspace \( V = \bigoplus_{i=1}^{\infty} V_i \) with dimensions \( \dim V_i = |t_i| \) and superdimensions \( \Dim V_i = t_i \in \mathbb{Z} \). Let \( \mathcal{L} \) be the free Lie superalgebra generated by \( V \). Then, \( \mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N \) and the subspaces \( \mathcal{L}_N \) have dimension given by

\[
\Dim \mathcal{L}_N = \sum_{g \mid N} \mu(g) \frac{W \left( \frac{N}{g} \right)}{g}
\]  \hspace{1cm} (2.11)
where \( g \) ranges over all common divisors of \( N \),
\[
W(N) = \sum_{s \in S(N)} \frac{(|s| - 1)!}{s!} \prod t(i)^{s_i}
\]  
(2.12)
with \( S(N) \) as in (2.10), is called the Witt partition function. Furthermore,
\[
\prod_{N=1}^{\infty} (1 - z^N)^{\pm \text{Dim} \mathcal{L}_N} = 1 + \sum_{i=1}^{\infty} f_\pm(i)z^i.
\]  
(2.13)
where \( f_+(i) = t(i) \) and \( f_-(i) = \text{dim} \mathcal{U}(\mathcal{L})_i \) is the dimension of the \( i \)-th homogeneous subspace of the universal enveloping algebra \( \mathcal{U}(\mathcal{L}) \). Identity (2.13) (the + case) is the generalized Witt identity.

Apply this interpretation to the determinant \( \det(1 - zT) \) which is a polynomial of degree \( 2|E| \) in the formal variable \( z \). It can be taken as a power series with coefficients \( t_i = 0 \), for \( i > 2|E| \). Comparison of (2.11), (2.12), (2.13) with (2.8), (2.9), (2.10) implies the following result:

**Theorem 2.5.** Given a graph \( G \), its edge matrix, let \( V = \bigoplus_{i=1}^{2|E|} V_i \) be a \( \mathbb{Z}_{>0} \)-graded superspace with finite dimensions \( \text{dim} V_i = |c_+(i)| \) and the superdimensions \( \text{Dim} V_i = c_+(i) \) given by (2.9), the coefficients of \( \det(1 - zT) \). Let \( \mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N \) be the free Lie superalgebra generated by \( V \). Then, \( \mathcal{L}_N \) has superdimension \( \text{Dim} \mathcal{L}_N = \Omega(N,T) \). The algebra has generalized Witt identity given by (1.2) and \( \zeta(z) \) is the generating function for the dimensions of the subspaces of the enveloping algebra \( \mathcal{U}(\mathcal{L}) \) which are \( \text{Dim} \mathcal{U}(\mathcal{L})_n = c_-(n), c_-(n) \) given by (2.9).

**Example 1.** \( G_1 \), the graph with 2 edges counterclockwisely oriented and hooked to a single vertex. The edge matrix for \( G_1 \) is the \( 4 \times 4 \) symmetric matrix
\[
T_{G_1} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}
\]
where \( A \) is the \( 2 \times 2 \) matrix with all entries equal to 1 and \( B \) is the \( 2 \times 2 \) matrix with the main diagonal entries equal to 0 and all the other entries equal to 1. In this case,
\[
\text{Tr} T_{G_1}^N = 2 + (-1)^N + 3^N,
\]
\[
\det(1 - zT_{G_1}) = 1 - 4z + 2z^2 + 4z^3 - 3z^4,
\]
so that the number of classes of reduced nonperiodic cycles of length \( N \) is given by the formula
\[
\Omega(N,T_{G_1}) = \frac{1}{N} \sum_{g \mid N} \mu(g) \left(2 + (-1)^N + 3^N\right)
\]
Let \( V = \bigoplus_{i=1}^{4} V_i \) be a \( \mathbb{Z}_{>0} \)-graded superspace with dimensions \( \text{dim} V_1 = 4, \text{dim} V_2 = 2, \text{dim} V_3 = 4, \text{dim} V_4 = 3 \) and superdimensions \( \text{Dim} V_1 = -4, \text{Dim} V_2 = 2, \text{Dim} V_3 = 4, \text{Dim} V_4 = -3 \). Let \( \mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N \) be the free graded Lie super algebra generated by \( V \). The dimension of \( \mathcal{L}_N \) is \( \text{Dim} \mathcal{L}_N = \Omega(N,T_{G_1}) \) which satisfies the identity
\[
\prod_{N=1}^{+\infty} (1 - z^N)^{\Omega(N,T_{G_1})} = 1 - 4z + 2z^2 + 4z^3 - 3z^4
\]
The enveloping algebra subspaces have dimensions given by the zeta function of $G_1$,

$$\prod_{N=1}^{+\infty} (1 - z^N)^{-\Omega(N,T_{G_1})} = 1 + \frac{1}{16} \sum_{n=1}^{\infty} ((-1)^n + 39 \cdot 3^n - 24 - 12n)z^n$$

**Example 2.** $G_2$, the bipartite graph with two vertices linked by three edges likewise oriented from one vertex to the other. The edge matrix of $G_2$ is as before but $A$ has all entries equal to zero and $B$ is

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The edge matrix has the trace $TrT^N_{G_2} = 0$ if $N$ is odd and $TrT^N_{G_2} = 4 + 2 \cdot 2^N$ if $N$ is even, and the determinant

$$det(1 - zT_{G_2}) = 1 - 6z^2 + 9z^4 - 4z^6$$

If $N$ is odd, the number of classes of nonperiodic cycles of length $N$ is $\Omega(N,T_{G_2}) = 0$, if $N$ is odd; if $N$ is even,

$$\Omega(N,T_{G_2}) = \frac{1}{N} \sum_{g|N} \mu(g)TrT^N_{G_2}$$

Let $V = \bigoplus_{i=1}^{3} V_{2i}$ be a $\mathbb{Z}_{>0}$-graded superspace with dimensions $dimV_2 = 6$, $dimV_4 = 9$, $dimV_6 = 4$ and superdimensions $DimV_2 = 6$, $DimV_4 = -9$, $DimV_6 = 4$. Let $\mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N$ be the free graded Lie superalgebra generated by $V$. The dimension of $\mathcal{L}_N$ is zero, for $N$ odd; for $N$ even, $Dim\mathcal{L}_N = \Omega(N,T_{G_2})$. The dimensions satisfy the identity

$$\prod_{N=1}^{+\infty} (1 - z^N)^{-\Omega(N,T_{G_2})} = 1 - 6z^2 + 9z^4 - 4z^6$$

The enveloping algebra subspaces have dimensions given by the zeta function of $G_2$,

$$\prod_{N=1}^{+\infty} (1 - z^N)^{-\Omega(N,T_{G_2})} = 1 + \frac{1}{18} \sum_{n=1}^{\infty} (2^{2n+5} - 6n - 14)z^{2n}$$

**References**


