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# Zeta function of a graph revisited

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**Abstract**

This talk reviews some new combinatorial and algebraic aspects of the zeta function of a finite graph.

## 1 Introduction

The $\zeta_G(z)$ function (also called the Ihara zeta function) of a finite connected and oriented graph $G$ (the graph may have multiple edges and loops) is formally defined by

$$\zeta_G(z) = \prod_{[p]} (1 - z^{N(p)})^{-1} \tag{1.1}$$

where the product is over the equivalence classes of non periodic backtrack-less and tail-less closed paths (cycles, for short) in $G$, $N(p)$ the length of a cycle in $[p]$. It can be rewritten as

$$\zeta_G(z)^{-1} = \prod_{N=1}^{+\infty} (1 - z^N)^{\Omega(N,T)} = det(I - zT), \tag{1.2}$$

$$\Omega(N,T) = \frac{1}{N} \sum_{g|N} \mu(g) Tr T^{\frac{N}{g}}. \tag{1.3}$$

where $N$ is a positive integer, $\mu$ the Möbius function: a) $\mu(+1) = +1$, b) $\mu(g) = 0$, if $g = p_1^{e_1}...p_q^{e_q}$, $p_1, ..., p_q$ primes, and any $e_i > 1$, c) $\mu(p_1...p_q) = (-1)^q$. $\Omega(N,T)$ is the number of equivalence classes of cycles of length $N$ in $G$, $T$ is the edge adjacency matrix of $G$, $g$ ranges over the positive divisors of $N$. A great deal of work has been done on this function. I refer the reader to [7], [8] and [9] for a comprehensive overview and references therein.

There is a remarkable resemblance between relations (1.2) and (1.3) with the famous *Witt identity* and the *Witt formula*, respectively, which are the following:

$$\prod_{N=1}^{+\infty} (1 - z^N)^{M(N,R)} = 1 - Rz, \tag{1.4}$$

$$M(N,R) = \frac{1}{N} \sum_{g|N} \mu(g) R^{\frac{N}{g}}. \tag{1.5}$$

 

where $R$ is a real number. See the introduction of [4] for a nice account about the Witt relations. The Witt formula is also called the *necklace polynomial* because it solves the problem of counting inequivalent nonperiodic colorings of a necklace with $N$ beads with at most $R$ colors. See [10]. Consider the graph with a single vertex and $R$ loops hooked to it and with the edges directed, say, counter clockwise. Witt formula gives the number of classes of equivalence of cycles of length $N$ that traverse the graph counter clockwise [6]. Another interpretation of the Witt formula is as a dimension formula for the homogenous subspaces of a graded free Lie algebra. The Poincaré-Witt-Birkoff theorem then implies the Witt identity. See [5]. In [10] Metropolis and Rota proved that the Witt polynomials satisfy several identities which they used to prove several important results about the *Necklace Algebra*, *Witt vectors*, etc. Natural questions to ask are: Relations (1.2) and (1.3) can be related to a coloring problem? Is there a connection with free Lie algebras? Do the $\Omega$'s satisfy some identities? It turns out that all these questions have positive answers as I have shown in [1]. These are new aspects of the zeta function of a graph some of which are reviewed in this talk.

In section 2.1 I show that the $\Omega$'s satisfy several identities analogous to those satisfied by the Witt polynomials. In section 2.2 $\Omega$ and the coefficients of the polynomial in $z$ given by $det(1-zT)$ are interpreted as the dimensions of certain vector spaces associated to a free Lie superalgebra. See [1] for the details. The results were obtained using ideas from [2], [3], [7] and [10].

## 2 Results

### 2.1 Some identities

In [10] Metropolis and Rota proved that the Witt polynomials satisfy several important identities which they used to build the necklace algebra. Theorem 2.1 shows that $\Omega$ satisfies similar identities. Theorem 2.2 gives a generalization of the classical *Strehl identity* [11].

**Theorem 2.1.** *Given the matrices $T_1$ and $T_2$ denote by $T_1 \otimes T_2$ the Kronecker product of $T_1$ and $T_2$. Then,*

$$\sum_{[s,t]=N} (s,t)\Omega(s,T_1)\Omega(t,T_2) = \Omega(N, T_1 \otimes T_2), \tag{2.1}$$

*$(s,t)$ is the maximum common divisor of $s$ and $t$. The summation is over the set of all positive integers $s,t$ such that $[s,t]=N$, $[s,t]$ the least common multiple of $s,t$. Also,*

$$\Omega(N, T^l) = \sum_{[l,t]=Nl} \frac{t}{N}\Omega(t,T). \tag{2.2}$$

*and*

$$(r,s)\Omega(N, T_1^{s/(r,s)} \otimes T_2^{r/(r,s)}) = \sum (rp,sq)\Omega(p,T_1)\Omega(q,T_2) \tag{2.3}$$

*The sum is over $p,q$ such that $pq/(pr,qs) = N/(r,s)$.*

 

**Theorem 2.2.**

$$\prod_{k\geq 1}\left[\frac{1}{det(1-z^kT_1)}\right]^{\Omega(k,T_2)}=\prod_{j\geq 1}\left[\frac{1}{det(1-z^jT_2)}\right]^{\Omega(j,T_1)} \tag{2.4}$$

It follows from the latter result that

$$\prod_{k\geq 1}\left[\zeta_{G_1}(z^k)\right]^{\Omega(k,T_2)}=\prod_{j\geq 1}\left[\zeta_{G_2}(z^j)\right]^{\Omega(j,T_1)} \tag{2.5}$$

**Theorem 2.3.**

$$\prod_{n\geq 1}\frac{1}{1-\Omega(n,T)z^n}=\left(\frac{1}{1-z}\right)^{\alpha} \tag{2.6}$$

$$\alpha=\sum_{d|n} d\left[\Omega(d,T)\right]^{n/d} \tag{2.7}$$

## 2.2 Dimension formulas

Theorem (2.4) below together with results from [3] imply that (1.2) and (1.3) can naturally be interpreted as data associated to a Lie superalgebra.

**Theorem 2.4.** *Define* $g(z) := \sum_{N=1}^{\infty}\frac{TrT^N}{N}z^N$. *Then,*

$$\prod_{N=1}^{+\infty}(1-z^N)^{\pm\Omega(N,T)}=e^{\mp g(z)}=[det(1-zT)]^{\pm}=1\mp\sum_{i=1}^{+\infty}c_{\pm}(i)z^i, \tag{2.8}$$

*where*

$$c_{\pm}(i)=\sum_{m=1}^{i}\lambda_{\pm}(m)\sum_{\substack{a_1+2a_2+...+ia_i=i\\ a_1+...+a_i=m}}\prod_{k=1}^{i}\frac{(TrT^k)^{a_k}}{a_k!k^{a_k}} \tag{2.9}$$

*with* $\lambda_+(m)=(-1)^{m+1}$, $\lambda_-(m)=+1$, $c_+(i)=0$ *for* $i>2|E|$, *and* $c_-(i)\geq 0$. *Furthemore,*

$$TrT^N=N\sum_{s\in S(N)}(\pm 1)^{|s|+1}\frac{(\mid s\mid -1)!}{s!}\prod c_{\pm}(i)^{s_i} \tag{2.10}$$

*where* $S(N)=\{s=(s_i)_{i\geq 1}\mid s_i\in\mathbb{Z}_{\geq 0},\sum is_i=N\}$ *and* $\mid s\mid=\sum s_i$, $s!=\prod s_i!$.

In section 2.3 of [3], given a formal power series $\sum_{i=1}^{+\infty}t_iz^i$ with $t_i\in\mathbb{Z}$, for all $i\geq 1$, the coefficients in the series are interpreted as the superdimensions of a $\mathbb{Z}_{>0}$-graded superspace $V=\bigoplus_{i=1}^{\infty}V_i$ with dimensions $dimV_i=|t_i|$ and superdimensions $DimV_i=t_i\in\mathbb{Z}$. Let $\mathcal{L}$ be the free Lie superalgebra generated by $V$. Then, $\mathcal{L}=\bigoplus_{N=1}^{\infty}\mathcal{L}_N$ and the subspaces $\mathcal{L}_N$ have dimension given by

$$Dim\mathcal{L}_N=\sum_{g|N}\frac{\mu(g)}{g}W\left(\frac{N}{g}\right) \tag{2.11}$$

 

where $g$ ranges over all common divisors of $N$,

$$W(N) = \sum_{s \in S(N)} \frac{(|\ s\ | - 1)!}{s!} \prod t(i)^{s_i} \tag{2.12}$$

with $S(N)$ as in (2.10), is called the *Witt partition function*. Furthermore,

$$\prod_{N=1}^{\infty} (1 - z^N)^{\pm Dim\mathcal{L}_N} = 1 \mp \sum_{i=1}^{\infty} f_{\pm}(i) z^i. \tag{2.13}$$

where $f_+(i) = t(i)$ and $f_-(i) = dim U(\mathcal{L})_i$ is the dimension of the $i$-th homogeneous subspace of the universal enveloping algebra $U(\mathcal{L})$. Identity (2.13) (the + case) is the *generalized Witt identity*.

Apply this interpretation to the determinant $det(1 - zT)$ which is a polynomial of degree $2|E|$ in the formal variable $z$. It can be taken as a power series with coefficients $t_i = 0$, for $i > 2|E|$. Comparison of (2.11), (2.12), (2.13) with (2.8), (2.9), (2.10) implies the following result:

**Theorem 2.5.** *Given a graph $G$, $T$ its edge matrix, let $V = \bigoplus_{i=1}^{2|E|} V_i$ be a $\mathbb{Z}_{>0}$-graded superspace with finite dimensions $dimV_i = |c_+(i)|$ and the superdimensions $DimV_i = c_+(i)$ given by (2.9), the coefficients of $det(1 - zT)$. Let $\mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N$ be the free Lie superalgebra generated by $V$. Then, $\mathcal{L}_N$ has superdimension $Dim\mathcal{L}_N = \Omega(N, T)$. The algebra has generalized Witt identity given by (1.2) and $\zeta(z)$ is the generating function for the dimensions of the subspaces of the enveloping algebra $U(\mathcal{L})$ which are $DimU(\mathcal{L})_n = c_-(n)$, $c_-(n)$ given by (2.9).*

**Example 1.** $G_1$, the graph with 2 edges counterclockwisely oriented and hooked to a single vertex. The edge matrix for $G_1$ is the $4 \times 4$ symmetric matrix

$$T_{G_1} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where $A$ is the $2 \times 2$ matrix with all entries equal to 1 and $B$ is the $2 \times 2$ matrix with the main diagonal entries equal to 0 and all the other entries equal to 1. In this case,

$$TrT_{G_1}^N = 2 + (-1)^N + 3^N, \quad det(1 - zT_{G_1}) = 1 - 4z + 2z^2 + 4z^3 - 3z^4$$

so that the number of classes of reduced nonperiodic cycles of length $N$ is given by the formula

$$\Omega(N, T_{G_1}) = \frac{1}{N} \sum_{g|N} \mu(g) \left( 2 + (-1)^{\frac{N}{g}} + 3^{\frac{N}{g}} \right)$$

Let $V = \bigoplus_{i=1}^{4} V_i$ be a $\mathbb{Z}_{>0}$-graded supespace with dimensions $dimV_1 = 4$, $dimV_2 = 2$, $dimV_3 = 4$, $dimV_4 = 3$ and superdimensions $DimV_1 = -4$, $DimV_2 = 2$, $DimV_3 = 4$, $DimV_4 = -3$. Let $\mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N$ be the free graded Lie super algebra generated by $V$. The dimension of $\mathcal{L}_N$ is $Dim\mathcal{L}_N = \Omega(N, T_{G_1})$ which satisfies the identity

$$\prod_{N=1}^{+\infty} (1 - z^N)^{\Omega(N, T_{G_1})} = 1 - 4z + 2z^2 + 4z^3 - 3z^4$$

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The enveloping algebra subspaces have dimensions given by the zeta function of $G_1$,

$$\prod_{N=1}^{+\infty}(1-z^N)^{-\Omega(N,T_{G_1})} \quad = \quad 1+\frac{1}{16}\sum_{n=1}^{\infty}((-1)^n+39\cdot 3^n-24-12n)z^n$$

**Example 2.** $G_2$, the bipartite graph with two vertices linked by three edges likewisely oriented from one vertex to the other. The edge matrix of $G_2$ is as before but $A$ has all entries equal to zero and $B$ is

$$B=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The edge matrix has the trace $TrT_{G_2}^N=0$ if $N$ is odd and $TrT_{G_2}^N=4+2\cdot 2^N$ if $N$ is even, and the determinant

$$det(1-zT_{G_2})=1-6z^2+9z^4-4z^6$$

If $N$ is odd, the number of classes of nonperiodic cycles of length $N$ is $\Omega(N,T_{G_2})=0$, if $N$ is odd; if $N$ is even,

$$\Omega(N,T_{G_2})=\frac{1}{N}\sum_{g|N}\mu(g)TrT_{G_2}^{\frac{N}{g}}$$

Let $V=\bigoplus_{i=1}^{3}V_{2i}$ be a $\mathbb{Z}_{>0}$-graded superspace with dimensions $dimV_2=6$, $dimV_4=9$, $dimV_6=4$ and superdimensions $DimV_2=6$, $DimV_4=-9$, $DimV_6=4$. Let $\mathcal{L}=\bigoplus_{N=1}^{\infty}\mathcal{L}_N$ be the free graded Lie superalgebra generated by $V$. The dimension of $\mathcal{L}_N$ is zero, for $N$ odd; for $N$ even, $Dim\mathcal{L}_N=\Omega(N,T_{G_2})$. The dimensions satisfy the identity

$$\prod_{N=1}^{+\infty}(1-z^N)^{\Omega(N,T_{G_2})} \quad = \quad 1-6z^2+9z^4-4z^6$$

The enveloping algebra subspaces have dimensions given by the zeta function of $G_2$,

$$\prod_{N=1}^{+\infty}(1-z^N)^{-\Omega(N,T_{G_2})} \quad = \quad 1+\frac{1}{18}\sum_{n=1}^{\infty}(2^{2n+5}-6n-14)z^{2n}$$

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