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Mountain Pass Algorithm Via Pohozaev Manifold

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1 Introduction

The celebrated *Mountain Pass Theorem* of Ambrosetti and Rabinowitz [1] has been widely used in the past forty years for finding weak solutions of a semilinear elliptic problem as critical points of an associated functional, and solutions are found on the minimax levels of the functional. A numerical approach of this theorem was first implemented by Choi and McKenna [2].

Later, Chen, Ni and Zhou [3] created a new algorithm based on the fact that the minimum of the associated functional constrained to the Nehari manifold is equal to the min-max level obtained by the Mountain Pass Theorem. This is true when the nonlinear terms in the equation are superquadratic, but not true in general for the asymptotically linear. However, more recently, the ground state level was shown to be equal to the minimum of the functional restricted to the Pohozaev manifold (Jeanjean and Tanaka [4]).

Our new algorithm is based in this analytical result. We numerically obtain positive solutions for an asymptotically linear problem using the well known important fact proved by Pohozaev that any weak solution of an elliptic equation of type

$$\begin{cases} - \quad \Delta u = g(u) \ \text{ in } \ \mathbb{R}^N, \\ \quad u \quad \in \quad H^1(\mathbb{R}^N), \end{cases} \tag{1}$$

must satisfy the Pohozaev identity, where $G(s) = \int_0^s g(t)dt$.

2 Main Results

We consider the semilinear elliptic problem

$$\begin{cases} -\Delta u \quad + \quad \lambda u = f(u) \ \text{ in } \ \mathbb{R}^N \\ \qquad\qquad u \ \in \qquad H^1(\mathbb{R}^N) \end{cases} \tag{2}$$

where $N \geq 2$ and λ is a positive constant. The associated functional I to this problem is well defined in $H^1(\mathbb{R}^N)$ and $I \in C^1(H^1(\mathbb{R})^N, \mathbb{R})$.

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Weak solutions u of problem (2) are precisely the critical points of I, i.e. $I'(u) = 0$, and any solution of (2) satisfies Pohozaev identity.

In this work, we present several lemmas which support the construction of the proposed algorithm. Among them, we have that, under some suitable conditions, for each function in $H^1(\mathbb{R}^N)$ there exists a unique projection on the Pohozaev manifold, and that a function in $H^1(\mathbb{R}^N)$ is a critical point of I if and only if it is a critical point of I restricted to the Pohozaev manifold $\mathcal{P}$.

3 Mountain Pass algorithm using Pohozaev manifold

Step 1. Take an initial guess $w_0 \in H^1(\mathbb{R}^N)$ such that $w_0 \neq 0$ and $\int G(w_0) > 0$;
Step 2 Find t_* such that $I(w_o(\frac{\cdot}{t_*})) = \max\ I(w_0(\frac{\cdot}{t}))$, $t > 0$, and set $w_1 = w_0(\frac{\cdot}{t_*})$;
Step 3 Find the steepest descent direction $\hat{v} \in H^1(\mathbb{R}^N)$ such that $[I(w_1 + \epsilon\hat{v}) - I(w_1)]/\epsilon$ is as negative as possible as $\epsilon \to 0$, obtaining $\hat{v} = -I'(w_1)$. If $||\hat{v}|| < \tau$, where τ is the estimator for convergence, then output and stop. Else, go to the next step;
Step 4. For α_0 small, there exists $t(\alpha_0)$ such that $(w_1 + \alpha_0\hat{v})\left(\frac{\cdot}{t(\alpha_0)}\right) \in \mathcal{P}$. Iterate $\alpha_k := k\alpha_0$, for $k \in \mathbb{Z}$ $(w_1 + \alpha_k\hat{v})\left(\frac{\cdot}{t(\alpha_k)}\right) \in \mathcal{P}$. Find $\hat{\alpha}$ such that

$$I\left((w_1 + \hat{\alpha}\hat{v})\left(\frac{\cdot}{t(\hat{\alpha})}\right)\right) = \min_{\alpha_k} I\left((w_1 + \alpha_k\hat{v})\left(\frac{\cdot}{t(\alpha_k)}\right)\right).$$

Step 5. Redefine $w_0 := w_1 + \hat{\alpha}\hat{v}$. Go to Step 2.

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