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On the shooting algorithm for partially-affine optimal control problems with control constraints

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Consider the following optimal control problem

$$\begin{aligned} \underset{(u,v)\in\mathcal{U}\times\mathcal{V}}{\text{minimize}} \quad & \phi(x(0),x(T)) \\ \text{subject to} \quad & \dot{x} = f_0(x,u) + \sum_{i=1}^{m} v_i f_i(x,u), \quad \text{a.e. on } [0,T] \\ & 0 = \eta_j(x(0),x(T)), \qquad \text{for } j = 1,\cdots,d_\eta. \end{aligned} \tag{1}$$

where the state x is governed by an ODE determined by the functions $f_0, f_i \colon \mathbb{R}^n \to \mathbb{R}^n$ that is *partially-affine in the controls*, this is, it has two types of controls: v that enters linearly and u, nonlinearly; the functions $\eta_j \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ describe the constraints in the endpoints of x; the controls u and v are taken in the Lebesgue spaces $\mathcal{U} := L^\infty([0,T];\mathbb{R}^l)$ and $\mathcal{V} := L^\infty([0,T];\mathbb{R}^m)$, respectively; and $\phi \colon \mathbb{R}^{2n} \to \mathbb{R}$ is the cost function to be minimized.

We propose a shooting scheme to numerically approximate the optimal solutions of problem (1) satisfying the additional control constraints

$$v(t) \in [0,1]^m, \ u(t) \in U, \quad \text{a.e. on } [0,T], \tag{2}$$

where $U \subseteq \mathbb{R}^l$ is open. We assume that the optimal solution $(\hat{x},\hat{u},\hat{v})$ is such that $\hat{v}$ is a finite concatenation of *bang* and *singular arcs*. More precisely, we suppose that the interval $[0,T]$ can be partitioned into finitely many subintervals of extremals $0 = T_0 < \cdots < T_N = T$, such that in each $[T_{k-1},T_k]$, each component $\hat{v}_i$ is either in the boundary of $[0,1]$ or in its interior. The times T_k are usually called *switching times*. When this structure occurs, it induces a transformation of (1)-(2) into a problem without control constraints. This transformation consists in defining a new problem in the normalized time interval $[0,1]$, that associates an unconstrained control variable to each singular arc of each $\hat{v}_i$, and a state variable with zero dynamics to each switching time T_k, for $k = 1,\dots,N-1$. This new *transformed problem* has as many controls as singular arcs of $\hat{v}$. Let us use (TP) to denote it. We can prove that a solution of the control-constrained problem (1)-(2)

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which is locally optimal in an L^1-neighborhood can be transformed into a solution of the transformed problem (TP), locally optimal in an L^∞-neighborhood. Our task is then reduced to solving an optimal control problem with no control constraints such as (1).

The *Pontryagin's Maximum Principle* applied to (1) yields the existence of a multiplier $\lambda = (\beta, p(\cdot))$ associated to the optimal solution $(\hat{x}, \hat{u}, \hat{v})$ such that, setting

$$\begin{aligned} H[\lambda](x,u,v) &:= p \cdot \left(f_0(x,u) + \sum_{i=1}^{m} v_i f_i(x,u) \right), \\ l[\lambda](x_0,x_T) &:= \phi(x_0,x_T) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(x_0,x_T), \end{aligned} \tag{3}$$

the pair $(\hat{x}, \hat{u}, \hat{v}), \lambda$ satisfies:

$$-\dot{p} = D_x H[\lambda](\hat{x},\hat{u},\hat{v}), \quad p(0) = -D_{x_0} l[\lambda](\hat{x}(0),\hat{x}(T)), \quad p(T) = D_{x_T} l[\lambda](\hat{x}(0),\hat{x}(T)), \tag{4}$$

$$D_{(u,v)} H[\lambda](\hat{x},\hat{u},\hat{v}) = 0. \tag{5}$$

We assume that the optimal control $(\hat{u}, \hat{v})$ can be written as a function of $\hat{x}$ and $\hat{p}$ from the stationarity condition (5), i.e.

$$\hat{u} = U(\hat{x},\hat{p}), \quad \hat{v} = V(\hat{x},\hat{p}). \tag{6}$$

Using this feedback representation[3] in the state equation and in (4), we get a boundary value problem for $(\hat{x}, \hat{p})$ with conditions both in $t = 0$ and in $t = T$, this is, a *two-point boundary value problem (TPBVP)*. The *shooting algorithm* that we propose (see also [1]) consists in solving this TPBVP by formulating it as a problem of finding the zeros of an appropriate function, called *shooting function*. We prove that the convergence of this shooting scheme follows from *second order sufficient optimality conditions* obtained in [3].

This work is an extension of the results present in [2], where the convergence was proved for problems in which all the controls appear linearly. The need for this extension comes from applications such as the optimization of aircraft trajectories, as in e.g. [4].

References

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[3] This feedback representation can be achieved, for instance, when the second order condition called *generalized Legendre-Clebsch condition* holds. See [1, 2].