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Duality Theory in Continuous-Time Linear Optimization

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Abstract. In this work, classical results from duality theory are obtained for continuous-time linear optimization problems with inequality constraints. Weak and strong duality properties, as well as, the complementary slackness theorem are established.

Keywords. Continuous-Time Optimization, Linear Optimization, Duality.

1 Introduction

This work is concerned with the continuous-time linear optimization problem posed as follows:

$$\begin{array}{ll} \text{minimize} & F(z) = \displaystyle\int_0^T c(t)^\top z(t) dt \\ \text{subject to} & A(t)z(t) \leq b(t) \text{ a.e. in } [0,T], \\ & z \in L^\infty([0,T];\mathbb{R}^n), \end{array} \tag{CLP}$$

where $A : [0,T] \to \mathbb{R}^{m\times n}$, $b : [0,T] \to \mathbb{R}^m$ and $c : [0,T] \to \mathbb{R}^n$ have essentially bounded and measurable entries in $[0,T]$.

This class of problems was introduced in 1953 by Bellman [2] and since then the theory was considerably developed. Optimality conditions, as well as, duality results were obtained. At first, the matrix A and vectors b and c were not allowed to vary with t and the optimality conditions and duality results were proved to be valid under very strong assumptions. More general problems were then tackled and less restrictive assumptions on the problem data were made. In 1980, the article [5] was published by Reiland generalizing previous results encountered in the literature until then. After the paper by Reiland, Zalmai published a series of articles on continuous-time nonlinear programming involving necessary and sufficient optimality conditions and duality theory. See [7], for example. The assumptions made by Zalmai were less restrictive than those ones by Reiland. One of the main tools used by Zalmai was a generalization of the Gordan Transposition Theorem for the continuous-time context (see [6]). After the works by Zalmai, almost all the literature produced in this area made use of the Gordan's Theorem. However, later, this theorem was discovered to be not valid (see [1]). Since then, the paper by Reiland became one of the main references on continuous-time linear programming.

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In [5], Reiland proved classical duality theorems for continuous-time linear programming problems under some constraint qualifications. He also presented an example showing that constraint qualifications are essential on establishing such results. So, in the continuous-time context, the linearity of the problem data is not itself a constraint qualification, as it is the case on finite dimensions.

Here, we obtain some classical results from duality theory for (CLP), namely, weak and strong duality properties and the complementary slackness theorem. As pointed by Reiland, in the continuous-time context, even for linear problems, a constraint qualification was necessary. We propose a new constraint qualification which is less restrictive and simpler to be verified in comparison with the one used by Reiland. In fact, Reiland in [5], required (i) the validity of a Slater-type condition; (ii) that the kernel of a certain operator (between infinite dimensional spaces) has finite dimensions and that its range is closed; and (iii) that the closure (in L^∞) of the feasible directions cone coincides with the linearized feasible directions cone. The new constraint qualification proposed here stands only on a property of the cone generated by the rows of matrix $A(t)$. It is worth mentioning that our constraint qualification may be verified even when matrix $A(t)$ does not have full rank.

The work is organized in the following way. Next, some preliminaries are given. The main results are stated and proved in Section 3. Conclusion words are provided in the last section.

2 Preliminaries

Optimality conditions for (CLP) can be found, for example, in de Oliveira [3] and in Monte and de Oliveira [4]. Particularly, in [4] we can find optimality conditions for a more general case: problems with nonlinear equality and inequality constraints. However, the constraint qualifications used in [3] and [4] are different. In what follows, we reproduce the optimality conditions for (CLP) developed in [3].

The set of all feasible solutions of (CLP) will be denoted as Ω:

$$\Omega = \{z \in L^\infty([0,T];\mathbb{R}^n) : A(t)z(t) \leq b(t) \text{ a.e. in } [0,T]\}.$$

We say that $\bar{z} \in \Omega$ is an optimal solution of (CLP) if

$$F(\bar{z}) \leq F(z) \ \forall z \in \Omega.$$

We will denote $I = \{1, 2, \ldots, m\}$.

Let $\beta > 0$ be a small scalar. Given $\bar{z} \in \Omega$, we will denote by $I_\beta(t)$ the index set of the β-active constraints at instant t, that is,

$$I_\beta(t) = \{i \in I : -\beta \leq a_i(t)^\top \bar{z}(t) - b_i(t) \leq 0\},$$

where $a_i(t)^\top$ denotes the i-th row of matrix $A(t)$.

We will denote by $I_a(t)$ the index set of the active constraints at instant t, that is,

$$I_a(t) = \{i \in I : a_i(t)^\top \bar{z}(t) - b_i(t) = 0\}.$$

Given an indices subset $\mathcal{I} \subset I$ and $t \in [0,T]$, we will denote by $A^{\mathcal{I}}(t)$ the matrix obtained from $A(t)$ by removing the rows whose indices do not belong to $\mathcal{I}$. The cardinality of $\mathcal{I}$ will be denoted by $|\mathcal{I}|$.

The smallest positive singular value of $A^{\mathcal{I}}(t)$ will be denoted by $\sigma_{p(t)}(A^{\mathcal{I}}(t))$, where $p(t) = \text{rank}(A^{\mathcal{I}}(t))$.

The cone generated by the rows of $A^{\mathcal{I}}(t)$ will be denoted by $\text{cone}(A^{\mathcal{I}}(t))$.

Definition 2.1. *We say that the Regularity Condition (β-RC) is satisfied at $\bar{z} \in \Omega$ if for almost all $t \in [0,T]$ there exist an indices subset $I_\sharp(t) \subset I_a(t)$ and a constant $\hat{K} > 0$ such that*

(i) $\text{cone}(A^{I_\sharp(t)}(t)) = \text{cone}(A^{I_\beta(t)}(t))$ *and* $\text{rank}(A^{I_\sharp(t)}(t)) = |I_\sharp(t)| =: p(t)$*;*

(ii) $\min\{\sigma_{p(t)}(A^{I_a(t)}(t)), \sigma_{p(t)}(A^{I_\sharp(t)}(t))\} \geq \hat{K}$*;*

for some $\beta > 0$.

A sufficient condition to the validity of (β-RC)-(i) is that, for almost all t, the matrix $A^{I_\beta(t)}(t)$ has full rank. In this case it is enough to set $I_\sharp = I_\beta$.

Theorem 2.1 (de Oliveira [3])**.** *Let $\bar{z} \in \Omega$ be an optimal solution of (CLP). Assume that (β-RC) is satisfied at $\bar{z}$. Then, there exists $u \in L^\infty([0,T];\mathbb{R}^m)$ such that*

$$c(t) + A(t)^\top u(t) = 0 \;\; a.e. \;\; in \; [0,T], \tag{1}$$

$$u(t)^\top [A(t)\bar{z}(t) - b(t)] = 0 \;\; a.e. \;\; in \; [0,T], \tag{2}$$

$$u(t) \geq 0 \;\; a.e. \;\; in \; [0,T]. \tag{3}$$

Remark 2.1. *Given the linearity of the problem, the converse of Theorem 2.1 holds true, we mean, if there exists $u \in L^\infty([0,T];\mathbb{R}^m)$ such that conditions (1)-(3) are valid, then $\bar{z} \in \Omega$ is an optimal solution of (CLP).*

3 Duality results

The main results will be stated and proved in this section. Below we give the definition of the dual problem. Next, among others, weak and strong duality properties and the complementary slackness theorem are presented.

Associated to (CLP), we define the following dual problem:

$$\begin{array}{ll} \text{maximize} & G(z) = \displaystyle\int_0^T b(t)^\top w(t) dt \\ \text{subject to} & A(t)^\top w(t) = c(t) \text{ a.e. in } [0,T], \\ & w(t) \leq 0 \text{ a.e. in } [0,T], \\ & w \in L^\infty([0,T];\mathbb{R}^m). \end{array} \tag{CDP}$$

From now on, (CLP) will be referred to as the primal problem.

The set of all feasible dual solutions will be denoted by Θ, i.e.,

$$\Theta = \{w \in L^\infty([0,T];\mathbb{R}^m) : A(t)^\top w(t) = c(t), \; w(t) \leq 0 \text{ a.e. in } [0,T]\}.$$

We begin with the weak duality property.

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Theorem 3.1 (Weak Duality)**.** *Let $z \in \Omega$ and $w \in \Theta$ be feasible solutions of (CLP) and (CDP). Then, $G(w) \leq F(z)$.*

Proof. Since $z \in \Omega$ and $w \in \Theta$, we know that $b(t) \geq A(t)z(t)$ and $w(t) \leq 0$ a.e. in $[0,T]$. Therefore, $b(t)^\top w(t) \leq z(t)^\top A(t)^\top w(t)$ a.e. in $[0,T]$. By making use of the fact that $A(t)^\top w(t) = c(t)$ a.e. in $[0,T]$, once $w \in \Theta$, we have $b(t)^\top w(t) \leq z(t)^\top c(t)$ a.e. in $[0,T]$. Thus,

$$G(w) = \int_0^T b(t)^\top w(t)dt \leq \int_0^T c(t)^\top z(t)dt = F(z).$$

□

Theorem 3.2. *Assume that $\Omega \neq \emptyset$. If there exists a sequence $\{z^k\} \subset \Omega$ such that $F(z^k) \to -\infty$ when $k \to \infty$, then $\Theta = \emptyset$.*

Proof. Suppose that there exists a sequence $\{z^k\} \subset \Omega$ such that $F(z^k) \to -\infty$ but $\Theta \neq \emptyset$. Then there exists $w \in \Theta$, and by Theorem 3.1, $G(w) \leq F(z^k)$ for all k. This inequality contradicts the fact that $F(z^k) \to -\infty$ when $k \to \infty$. Therefore, $\Theta = \emptyset$. □

Theorem 3.3. *Assume that $\Theta \neq \emptyset$. If there exists a sequence $\{w^k\} \subset \Theta$ such that $G(w^k) \to +\infty$ when $k \to \infty$, then $\Omega = \emptyset$.*

Proof. Suppose that exists a sequence $\{w^k\} \subset \Theta$ such that $G(w^k) \to +\infty$ while $\Omega \neq \emptyset$. Then there exists $z \in \Omega$, and by Theorem 3.1, $G(w^k) \leq F(z)$ for all k. This inequality contradicts the fact that $G(w^k) \to +\infty$ when $k \to \infty$. Thus, $\Omega = \emptyset$. □

Lemma 3.1. *Let $\bar{z} \in \Omega$ be an optimal solution of (CLP). Assume that (β-RC) is satisfied at $\bar{z}$.Then $\tilde{w} = -u$, where $u \in L^\infty([0,T];\mathbb{R}^n)$ is the Lagrange multiplier associated to $\bar{z}$, given in Theorem 2.1, is a feasible solution of (CDP), that is, $\tilde{w} \in \Theta$.*

Proof. From (1), we have $c(t) = -A(t)^\top u(t)$ a.e. in $[0,T]$, so that $A(t)^\top \tilde{w}(t) = c(t)$ a.e. in $[0,T]$. From (3), it follows that $\tilde{w}(t) = -u(t) \leq 0$ a.e. in $[0,T]$. Therefore, $\tilde{w} \in \Theta$. □

Theorem 3.4. *Assume that $\Omega \neq \emptyset$ and that (β-RC) is satisfied at each $z \in \Omega$. If $\Theta = \emptyset$, then (CLP) does not have any optimal solution.*

Proof. If (CLP) has an optimal solution, by Lemma 3.1, there exists $\tilde{w} \in \Theta$, contradicting the hypothesis. □

Theorem 3.5. *Assume that $\Theta \neq \emptyset$ and that there exists a constant $K_A > 0$ such that $\det(A(t)^\top A(t)) \geq K_A$ a.e. in $[0,T]$. If $\Omega = \emptyset$, then (CDP) does not have any optimal solution.*

Proof. Suppose that (CDP) has an optimal solution, say $\bar{w} \in \Theta$. Let us note that the existence of a constant $K_A > 0$ such that $\det(A(t)^\top A(t)) \geq K_A$ a.e. in $[0,T]$ implies that the full rank assumption in Monte and de Oliveira [4] is satisfied. It follows from Theorem 4.1 in [4], that there exist $u \in L^\infty([0,T];\mathbb{R}^n)$ and $v \in L^\infty([0,T];\mathbb{R}^m)$ such that $b(t) + A(t)u(t) - v(t) = 0$, $v(t) \geq 0$ and $v(t)^\top \bar{w}(t) = 0$ a.e. in $[0,T]$. Taking $\tilde{z} = -u$, we have $A(t)\tilde{z}(t) = b(t) - v(t) \leq b(t)$ a.e. in $[0,T]$, that is, $\tilde{z} \in \Omega$, contradicting the hypothesis. □

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Theorem 3.6 (Strong Duality)**.** *Let $\bar{z} \in \Omega$ and $\bar{w} \in \Theta$ be feasible solutions of (CLP) and (CDP). Assume that (β-RC) is satisfied at $\bar{z}$. Then $F(\bar{z}) = G(\bar{w})$ if, and only if, $\bar{z}$ is an optimal solution of (CLP) and $\bar{w}$ is an optimal solution of (CDP).*

Proof. Suppose that $F(\bar{z}) = G(\bar{w})$. It follows from the weak duality property, Theorem 3.1, that $F(z) \geq G(\bar{w}) = F(\bar{z})$ for all $z \in \Omega$. Thus, $\bar{z}$ is an optimal solution of (CLP). It follows from the same property that $G(w) \leq F(\bar{z}) = G(\bar{w})$ for all $w \in \Theta$, so that $\bar{w}$ is an optimal solution of (CDP).

Reciprocally, suppose that $\bar{z}$ is an optimal solution of (CLP) and $\bar{w}$ is an optimal solution of (CDP). We know from Lemma 3.1 that $\tilde{w} = -u \in \Theta$, where u is the Lagrange multiplier associated to $\bar{z}$, given in Theorem 2.1. It follows from (2) that $u_i(t) = 0$ for $i \notin I_a(t)$ a.e. in $[0, T]$. Therefore, from (1) we have that $A^{I_a(t)}(t)^\top u^{I_a(t)}(t) = A(t)^\top u(t) = -c(t)$ a.e. in $[0, T]$. For $i \in I_a(t)$ we have $a_i(t)^\top \bar{z}(t) = b_i(t)$ a.e. in $[0, T]$. In vectorial notation, $A^{I_a(t)}(t)\bar{z}(t) = b^{I_a(t)}$ a.e. in $[0, T]$. Keeping this in mind, we obtain

$$\begin{aligned} G(\bar{w}) &\geq G(\tilde{w}) = \int_0^T b(t)^\top \tilde{w}(t)dt = -\int_0^T b(t)^\top u(t)dt \\ &= -\int_0^T b^{I_a(t)}(t)^\top u^{I_a(t)}(t)dt = -\int_0^T \bar{z}(t)^\top A^{I_a(t)}(t)^\top u^{I_a(t)}(t)dt \\ &= \int_0^T \bar{z}(t)^\top c(t)dt = F(\bar{z}). \end{aligned}$$

On the other hand, we have from the weak duality property that $G(\bar{w}) \leq F(\bar{z})$. Thus, $G(\bar{w}) = F(\bar{z})$. □

Observe in the proof of the last theorem that (β-RC) was only needed to prove that the optimal values of the primal and dual problems coincide. The other statement, which is a direct consequence of weak duality, always holds.

Corollary 3.1. *Let $\bar{z} \in \Omega$ be an optimal solution of (CLP). Assume that (β-RC) is satisfied at $\bar{z}$. Then $\tilde{w} = -u$, where $u \in L^\infty([0, T]; \mathbb{R}^n)$ is the Lagrange multiplier associated to $\bar{z}$, given in Theorem 2.1, is an optimal solution of (CDP) with $G(\tilde{w}) = F(\bar{z})$.*

Proof. We know from Lemma 3.1 that $\tilde{w} \in \Theta$ and we saw in the proof of Theorem 3.6 that $G(\tilde{w}) = F(\bar{z})$. It follows from Theorem 3.6 that $\tilde{w}$ is an optimal solution of (CDP). □

Theorem 3.7 (Complementary Slackness Theorem)**.** *Let $\bar{z} \in L^\infty([0, T]; \mathbb{R}^n)$ and $\bar{w} \in L^\infty([0, T]; \mathbb{R}^m)$. Assume that ($\beta$-RC) is satisfied at $\bar{z}$. Then, solutions $\bar{z} \in L^\infty([0, T]; \mathbb{R}^n)$ and $\bar{w} \in L^\infty([0, T]; \mathbb{R}^m)$ are optimal for (CLP) and (CDP) if, and only if,*

$$A(t)\bar{z}(t) + \bar{y}(t) = b(t), \ \bar{y}(t) \geq 0 \ a.e. \ in \ [0, T], \tag{4}$$

$$A(t)^\top \bar{w}(t) = c(t), \ \bar{w}(t) \leq 0 \ a.e. \ in \ [0, T], \tag{5}$$

$$\bar{y}(t)^\top \bar{w}(t) = 0 \ a.e. \ in \ [0, T]. \tag{6}$$

Proof. Suppose that $\bar{z} \in L^\infty([0, T]; \mathbb{R}^n)$ and $\bar{w} \in L^\infty([0, T]; \mathbb{R}^m)$ are optimal for (CLP) and (CDP). It is clear that $\bar{z} \in \Omega$ and that $\bar{w} \in \Theta$. Defining $\bar{y}(t) = b(t) - A(t)\bar{z}(t)$ a.e.

in $[0, T]$ and taking into account that $\bar{z} \in \Omega$, it follows that (4) holds. The validity of (5) follows directly from $\bar{w} \in \Theta$. If $\tilde{w} = -u$, where $u \in L^\infty([0, T]; \mathbb{R}^n)$ is the Lagrange multiplier associated to $\bar{z}$, we have from Corollary 3.1 that

$$\begin{aligned}\int_0^T \bar{y}(t)^\top \bar{w}(t)dt &= \int_0^T [b(t) - A(t)\bar{z}(t)]^\top \bar{w}(t)dt \\ &= \int_0^T b(t)^\top \bar{w}(t)dt - \int_0^T \bar{z}(t)^\top A(t)^\top \bar{w}(t)dt \\ &= \int_0^T b(t)^\top \bar{w}(t)dt - \int_0^T \bar{z}(t)^\top c(t)dt \\ &= G(\bar{w}) - F(\bar{z}) = G(\tilde{w}) - F(\bar{z}) = 0.\end{aligned}$$

As $\bar{y}(t)^\top \bar{w}(t) \leq 0$ a.e. in $[0, T]$, (6) holds.

Reciprocally, if (4)-(6) hold, it is clear that $\bar{z} \in \Omega$ and that $\bar{w} \in \Theta$. Moreover,

$$\begin{aligned}G(\bar{w}) &= \int_0^T b(t)^\top \bar{w}(t)dt = \int_0^T [A(t)\bar{z}(t) + \bar{y}(t)]^\top \bar{w}(t)dt \\ &= \int_0^T \bar{z}(t)^\top A(t)^\top \bar{w}(t)dt + \int_0^T \bar{y}(t)^\top \bar{w}(t)dt = \int_0^T \bar{z}(t)^\top c(t)dt \\ &= F(\bar{z}).\end{aligned}$$

By Theorem 3.6, we conclude that $\bar{z} \in \Omega$ and that $\bar{w} \in \Theta$ are optimal solutions of (CLP) and (CDP). □

Note that the assumption that (β-RC) is valid, in theorem above, is needed only to establish that conditions (4)-(6) are necessary when we have optimal solutions of primal and dual problems. The other statement holds without such an assumption.

Theorem 3.8. *If the primal problem (CLP) has an optimal solution in which (β-RC) is satisfied, then the dual problem (CDP) has an optimal solution.*

Proof. If (CLP) has an optimal solution, we know from Corollary 3.1 that (CDP) also has an optimal solution. □

Theorem 3.9. *Assume that there exists $K_A > 0$ such that $\det(A(t)^\top A(t)) \geq K_A$ a.e. in $[0, T]$. If the dual problem (CDP) has an optimal solution, then the primal problem (CLP) has an optimal solution.*

Proof. If (CDP) has an optimal solution, say $\bar{w}$, it follows from Theorem 4.1 in Monte and de Oliveira [4], that there exist $u \in L^\infty([0, T]; \mathbb{R}^n)$ and $v \in L^\infty([0, T]; \mathbb{R}^m)$ such that $b(t) + A(t)u(t) - v(t) = 0$, $v(t) \geq 0$ and $v(t)^\top \bar{w}(t) = 0$ a.e. in $[0, T]$ (the existence of $K_A > 0$ such that $\det(A(t)^\top A(t)) \geq K_A$ a.e. in $[0, T]$ implies that the full rank assumption in [4] is satisfied). The pair $(\bar{z}, \bar{y}) = (-u, v)$ satisfies (4). It is clear that $\bar{w}$ satisfies (5) and that $v(t)^\top \bar{w}(t) = 0$ a.e. in $[0, T]$ implies that (6) also holds. It follows from Theorem 3.7 (Complementary Slackness Theorem) that $\bar{z}$ is an optimal solution of (CLP). □

4 Conclusions

Duality theory for continuous-time linear optimization problems with inequality constraints was carried out. Classical duality results such as weak and strong duality properties and the complementary slackness theorem were established, where regularity conditions were necessary. Such regularity conditions are less restrictive than those encountered in the literature. Thus, this work gives theoretical contributions to the class of continuous-time optimization problems. As already mentioned in the introduction, such problems were introduced in the fifties by Bellman in [2] as, according to Bellman himself, an interesting and significant class of production and allocation problems, called "bottleneck problems".

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