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Stochastic Differential Equations driven by Fractional Brownian Motion with Markovian Switching[1]

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Abstract. Two circles of ideas permeate a great deal of the specialized literature on stochastic modelling nowadays: that which study long-range dependence phenomena, with particular emphasis on *fractional Brownian motion*, and the one which considers *stochastic differential equations with Markov switching*. In this paper we put together these topics by analysing a class of stochastic differential equations with Markov switching and subject to a fractional Brownian motion perturbation. We prove global existence and uniqueness results for this class of stochastic equations. This, in turn, set the stage for an avenue of research associated with this class of models which seems promising; particularly that associated with control problems.

Key words. Stochastic Differential Equation, Fractional Brownian motion, Markov Jump Linear Systems.

1 Introduction

In spite of the fact that classical stochastic control theory of dynamical systems driven by Brownian motions has been widely celebrated as a great achievement in stochastic analysis and of fundamental importance in applications, it has been realized that there are a wide range of applications in which the Brownian motion modelling of disturbance is not fully adequate. This happens, for instance, when the issue of *long-range dependence* (notice that Brownian motion has independent increments) comes to the fore in analysing the problem. In view of that, appropriate stochastic models for long-range dependent phenomena have attracted a great deal of interest in recent years. In particular, stochastic systems in which the driving process is a fractional Brownian motion (fBm) has been used.

The first systematic attempt to weave fBm into the stochastic process theory framework was made in [8]. Roughly speaking, fBm is a self-similar Gaussian processes with stationary increments (it is nonstationary as a whole). In fact, fBm is governed by a parameter H called *Hurst parameter*. It is the Hurst parameter that governs the self-similarity

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degree; the long-range dependence degree (the decaying rate of the autocorrelation coefficient function) and the irregularity of sample paths. It is a well-known result that for each H there exist a unique Gaussian process, self-similar, vanishing at the origin and having stationary increments. So, a fBm, which we denote by $B^H = \{B^H(t), t \in [0,T]\}$, is a continuous and centered Gaussian process with covariance function

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - \mid t - s \mid^{2H}),\ s,t \in \mathbb{R}_+, \tag{1}$$

where H is the Hurst parameter in $(0,1)$. When $H \geq 1/2$ the correlation is positive and we say that fBm has the property of long-range dependence. Throughout this paper it is assumed that $H \in (1/2, 1)$ is arbitrary but fixed.

The main technical hindrance in dealing with fBm is the fact that this process is not a semimartingale (except when is ordinary Brownian motion). In addition, it is not Markovian. Therefore, it is not possible, for instance, to make use of the powerful machinery of stochastic calculus, available in the current literature. Although it has not been possible to define up to now a general theory of stochastic integral for $H \in (0,1)$, it has been possible to advance in this regard for some sub-intervals. This subject has been investigated by a substantial number of authors and a host of recent new results has appeared in the specialized literature (see, for instance [1, 10]). This, in turn, has allowed various interesting mathematical applications in the context of fBm, with a particular bearing in the study of stochastic differential equations (SDE's) driven by fBm. For instance, a general class of SDE driven by an $n-$dimensional fBm B^H is the following:

$$x(t) = x(0) + \int_0^t b(s, x(s))ds + \int_0^t \sigma(s, x(s))dB^H(s) \tag{2}$$

where b, σ representing measurable random fields with appropriate dimensions. SDE's of this kind have been studied by many authors, see, for example, [3, 5, 11], to mention a few. Nudged by the significant body of results on stochastic integral for fBm, equation (2) have been used in the study of the stochastic control problem for dynamical systems with fBm perturbation. Although the results are of comparatively recent vintage and still meager, this avenue of research seems to be very encouraging and therefore has, in recent years, generated a flurry of interest. To some extent, the study has been focused in the case of linear dynamical systems driven by fBm and a quadratic cost functional. Initial work on this scenario was carried out, for instance, in [4].

In this paper, we are interested in multidimensional SDE's with Markovian switching of the integral form

$$x(t) = x(0) + \int_0^t b(x(s), s, \theta(s))ds + \int_0^t \sigma(x(s), s, \theta(s))dB^H(s) \tag{3}$$

where $\theta(t)$ is a Markov chain taking values on $\mathbb{S} = \{1, 2, \ldots, N\}$. This equation can also be seen as a set of N equations (for more details see [9]). For the case in which (3) is linear and the perturbation is modeled by the classical Brownian motion, this equation has been dubbed in the specialized literature as Markov jump linear systems (MJLS) and the coherent body of theoretical results on this subject makes it by now a full fledged theory (see, e.g., [2] and references therein). The theory of MJLS has contributed for the solution of a long standing challenge in control theory which has been to characterize mathematically uncertainty which is due to *abrupt changes*. This is a key issue to guarantee specific behaviors and stringent performances of dynamical systems which are subject to abrupt changes, such as failure.

In order to address adequately the problem of stochastic control for dynamical systems modeled by (3), see for e.g. [6], a key underlying issue is to ensure existence and uniqueness of solution for this stochastic equation. As far as the authors are aware, there is no available result on the literature on this regard. In view of this, we carry out in this paper a first systematic attempt to provide some preliminaries results on the issue of existence and uniqueness of solution for this equation. Due to the strongly non-Markovian nature of fBm, the most natural way to define the stochastic integral is to do it pathwise, for a.e. ω as in [11].

An outline of the content of this paper goes as follows. In section 2 we formulate our problem. In section 3 we provide the basic facts of the theory used in this paper. In section 4 we prove the main result on existence and uniqueness of solution for stochastic differential equations driven by a fBm with $H > 1/2$ and subject to Markovian switching.

2 Problem formulation

Let us fix an underlying complete probability space $(\Omega, \mathcal{F}, P)$ carrying the following independent processes:

1. A $\mathbb{R}^n-$valued zero mean standard fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ denoted by $B^H = \{B^H(t); t \in [0,T]\}$ such that $B^H(0) = 0$. The filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions.

2. An homogeneous Markov process $\theta = \{\theta(t), t \in [0,T]\}$ adapted to $\mathcal{F}_t$, with right continuous trajectories, taking values on the finite set $\mathbb{S} = \{1, 2, \ldots, N\}$ and with a stationary standard transition probability matrix function $\{P_\Delta(i,j)\}_{i,j\in\mathbb{S}}$ given by

$$P_\Delta(i,j) = P(\theta(t+\Delta) = j|\theta(t) = i) = \begin{cases} \lambda_{ij}\Delta + o_{ij}(\Delta), & i \neq j \\ 1 + \lambda_{ii}\Delta + o_{ii}(\Delta), & i = j \end{cases}$$

with $o(\Delta)$ denoting an infinitesimal of higher order than Δ, and with infinitesimal matrix $\Lambda = (\lambda_{ij})_{i,j\in\mathbb{S}}$, here $\lambda_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\lambda_{ii} = -\sum_{j\neq i} \lambda_{ij}$.

Equation (3) is equivalent to the following $n-$dimensional SDE with Markovian switching of the form

$$\begin{cases} dx(t) = b(t, x(t), \theta(t))dt + \sigma(t, x(t), \theta(t))dB^H(t) \\ x(0) = x_0 \in \mathbb{R}^n, \theta(0) = i, t \in [0,T],\ i = 1, \ldots, N, \end{cases} \tag{4}$$

where b and σ are measurable continuous functions wit $b(\cdot) : [0,T] \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$, and $\sigma(\cdot) : [0,T] \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n\times m}$.

To ensure existence and uniqueness of solutions for SDE (4) we impose the following assumptions which are supposed to hold $P-$almost surely for $\omega \in \Omega$, i.e., the inequalities are supposed to hold for $P-$ a.s. $\omega \in \Omega$.

Hypothesis ($\mathbf{H}_1$): The function $b(\cdot)$ is Lipschitz continuous and has linear growth in the variable x, uniformly in t, that is, there exists $L_1 > 0$ such that the following properties hold for each $j = 1, \ldots, n$ and for all $t \in [0, T]$:

$$\begin{cases} |b(t,x,i) - b(t,y,i)| \leq L_1|x-y|, \forall x, y \in R^n; \\ |b(t,x,i)| \leq L_2(|x|+1), \forall x \in \mathbb{R}^n. \end{cases}$$

Hypothesis ($\mathbf{H}_2$): The function $\sigma(\cdot)$ is Lipschitz continuous and continuously differentiable and with the bounded derivative satisfying a local Hölder continuity property in x, and Hölder continuous in time, that is, there exist some constants β, δ such that $0 < \beta, \delta \leq 1$ and for every $R \geq 0$ there exists $M_R > 0$ such that the following properties hold for each $i = 1, \ldots, n$ and for all $x, y \in \mathbb{R}^n$ and $s, t \in [0, T]$:

$$\begin{cases} |\sigma(t,x,i) - \sigma(s,y,i)| \leq M_0|x-y|; \\ |\partial_{x_j}\sigma(t,x,i)| \leq M_1; \\ |\partial_{x_j}\sigma(t,x,i) - \partial_{y_j}\sigma(t,y,i)| \leq M_R|x-y|^{\delta}, \forall |x|, |y| \leq R; \\ |\sigma(t,x,i) - \sigma(s,y,i)| + |\partial_{x_j}\sigma(t,x,i) - \partial_{x_j}\sigma(s,x,i)| \leq M_0|t-s|^{\beta}, \end{cases}$$

where ∂_{x_j} denotes the partial derivative of x with respect to variable x_j and the constants L_1, L_2, M_0, M_1 and M_R may depend on $\omega \in \Omega$.

3 Preliminaries

Let us consider the following normed spaces[4]. Fix $0 < \alpha < 1/2$ and denote by

1. $W_0^{\alpha,\infty}([0,T];\mathbb{R}^d)$, the space of measurable functions $f : [0,T] \to \mathbb{R}^d$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]} \left(|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty.$$

2. $C^{\alpha}(0,T;\mathbb{R}^d)$, for any $0 < \alpha \leq 1$, the space of $\alpha-$Hölder continuous functions $f : [0,T] \to \mathbb{R}^d$, equipped with the norm

$$\|f\|_{\alpha} := \|f\|_{\infty} + \sup_{0\leq s<t\leq T} \frac{\|f(t)-f(s)\|}{(t-s)^{\alpha}} < \infty, \text{ where } \|f\|_{\infty} := \sup_{t\in[0,T]} |f(t)|.$$

3. $W_T^{1-\alpha,\infty}(0,T)$, the space of measurable functions $g : [0,T] \to \mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty} := \sup_{0<s<t<T} \left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_0^t \frac{|g(s)-g(y)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

[4]These spaces were introduced by Nualart and Răşcanu [11]

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4. $W_0^{\alpha,1}(0,T)$, the space of measurable functions f on $[0,T]$ such that

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^\alpha} ds + \int_0^T \int_0^s \frac{|f(s)-f(y)|}{(s-y)^{\alpha+1}} dyds < \infty.$$

Remark 3.1. *Given any ϵ such that $0 < \epsilon < \alpha$, we have the following inclusions $C^{\alpha+\epsilon} \subseteq W_0^{\alpha,\infty} \subseteq C^{\alpha-\epsilon}$ and $C^{1-\alpha+\epsilon}(0,T) \subset W_T^{1-\alpha,\infty}(0,T) \subset C^{1-\alpha}(0,T), \forall \epsilon > 0$.*

Remark 3.2. *The trajectories of fractional Brownian motion B^H, for any $0 < \alpha < H$, belong to $C^\alpha(0,T;\mathbb{R}^d)$ almost surely. Therefore, by (3.1), we obtain that the trajectories of B^H for any $0 < \alpha < H$ belong to $W_0^{\alpha,\infty}$ almost surely.*

The following two propositions are generalized results of [11].

Proposition 3.1. *Fix $0 < \alpha < 1/2$. Let $F_t^{(b)}(f) = \int_0^t b(s, f(s), i)ds$ be a Lebesgue integral with b satisfying the assumptions ($\boldsymbol{H}_1$) with $\rho = \frac{1}{\alpha}$. If $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$ then $F_t^{(b)}(f) \in C^{1-\alpha}(0,T;\mathbb{R}^n) \subset W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$ and $\|F^{(b)}(f)\|_{1-\alpha} \leq C(1 + \|f\|_\infty)$, with a positive constant C depending on α, T and L_0.*

Proposition 3.2. *Let $f \in W_0^{\alpha,1}(0,T)$ be the integral $G_t^{(\sigma)}(f) = \int_0^t \sigma(s, f(s), i)\, dB_s^H$, with σ satisfying the assumptions ($\boldsymbol{H}_2$) with $\beta > \alpha$. If $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$, then $G^{(\sigma)}(f) \in C^{1-\alpha}(0,T;\mathbb{R}^n) \subset W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$ and $\|G^{(\sigma)}(f)\|_{1-\alpha} \leq D + M_0\|f\|_{\alpha,\infty}$, with a positive constant D depending on T, α, β and M_0.*

The next theorem, proved in [5], ensures the existence and pathwise uniqueness of a strong solution for (2).

Theorem 3.1. *Let b and σ be continuous functions. Such that b is Lipschitz continuous and linear growth in the second variable and uniformly in the first, and σ is continuously differentiable in x and with partial derivative being bounded and local Hölder continuous with parameter δ, and Hölder continuous in time with parameter β. Under these assumptions, if $1 - H < \alpha < \min\{\frac{1}{2}, \beta, \frac{\delta}{2}\}$ for $\delta, \beta \in (0,1]$, then there exists a unique strong solution of the stochastic equation (2).*

4 Existence and Uniqueness Result

Definition 4.1. *An $\mathbb{R}^n-$valued stochastic process $X = \{X(t); t \in [0,T]\}$ is called a solution of (4) if: X is $\mathcal{F}_t-$adapted; a.s. the trajectories of X belong to $W_0^{\alpha,\infty}([0,T];\mathbb{R}^n)$ and the equation (3) holds with probability 1. In particular, a solution $X = \{X(t); t \in [0,T]\}$ is said to be unique if any other solution $\hat{X} = \{\hat{X}(t); t \in [0,T]\}$ is indistinguishable from X.*

Theorem 4.1. *Assume that the coefficients b and σ satisfy the assumptions ($\boldsymbol{H}_1$) and ($\boldsymbol{H}_2$), respectively, with $\delta, \beta \in (0,1]$. If $1 - H < \alpha < \min\{\frac{1}{2}, \beta, \frac{\delta}{2}\}$, then there exists a unique solution $X \in W_0^{\alpha,\infty}(0,T)$ of SDE given by (3). Moreover the solution is Hölder continuous of order $1 - \alpha$.*

Proof. First, it is not difficult to prove that X is $\mathcal{F}_t-$adapted. Second, if $X \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$ is a solution of equation (3), then $X \in C^{1-\alpha}(0,T;\mathbb{R}^n)$. In fact, for all $u \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$ we have $F^{(b)}(u)$ and $G^{(\sigma)}(u)$ belong to $C^{1-\alpha}(0,T;\mathbb{R}^n)$ by Proposition 3.1 and Proposition 3.2, respectively. Hence

$$X = x_0 + F^{(b)}(X) + G^{(\sigma)}(X) \in C^{1-\alpha}(0,T;\mathbb{R}^n) \subset W_0^{\alpha,\infty}.$$

Then, since almost every sample path of the Markov process $\theta(\cdot)$ is right-continuous step function with a finite number of simple jumps on $[0,T]$, there is a sequence $\{\tau_\ell\}_{\ell\geq 0}$ of stopping times such that:

- for almost every $\omega \in \Omega$ there is a finite $\bar{\ell} = \bar{\ell}(\omega)$ for $0 = \tau_0 < \tau_1 < \cdots < \tau_{\bar{\ell}} = T$ and $\tau_\ell = T$ if $\ell > \bar{\ell}$;
- $\theta(t)$ is a random constant on every stochastic interval $[\tau_\ell, \tau_{\ell+1}[$, i.e. for every $\ell \geq 0$ $\theta(t) = \theta(\tau_\ell)$ on $\tau_\ell \leq t < \tau_{\ell+1}$.

We first consider (3) in the stochastic interval $[0,\tau_1[$ which becomes

$$x_t^{(1)} = x_0 + \int_0^t b(s,x_s,\theta_0)ds + \int_0^t \sigma(s,x_s,\theta_0)dB^H(s) \tag{5}$$

with initial conditions x_0 and θ_0. By Theorem 3.1 we can see that (5) has a unique solution $x_t^{(1)}$ which belongs to $W_0^{\alpha,\infty}(0,\tau_1;\mathbb{R}^n)$. Next, consider (3) on $t \in [\tau_1,\tau_2[$ which becomes

$$x_t^{(2)} = x_{\tau_1} + \int_{\tau_1}^t b(s,x_s,\theta(\tau_1))ds + \int_{\tau_1}^t \sigma(s,x_s,\theta(\tau_1))dB^H(s) \tag{6}$$

with initial data x_{τ_1} and θ_{τ_1}. Again, by Theorem 3.1, equation (6) has a unique solution $x_t^{(2)}$ which belongs to $W_0^{\alpha,\infty}(\tau_1,\tau_2;\mathbb{R}^n)$. Bearing in mind that the Markov process $\theta(t)$ has only a finite number of jumps in a finite time interval, we can repeat this procedure to obtain a unique solution, $X(t)$, for (3) in the interval $[0,T]$. Such solution have the form

$$X(t) = \begin{cases} x_t^{(1)}, & \text{if } t \in [0,\tau_1[\\ x_t^{(2)}, & \text{if } t \in [\tau_1,\tau_2[\\ \vdots & \\ x_t^{(\ell)}, & \text{if } t \in [\tau_{\ell-1},T]. \end{cases}$$

□

5 CONCLUSIONS

We have established conditions for existence and uniqueness of solution for a class of stochastic differential equations with Markov switching and subject to a fractional Brownian motion perturbation. We believe that these results are particularly important in the study of the control problem for this class of stochastic dynamical systems, since it guarantees existence and uniqueness of solution.

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