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A posteriori error estimates for enriched mixed formulations

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Abstract. The purpose of this paper is to derive efficient and robust a posteriori error estimations for the enriched mixed methods proposed in [2, 4]. The general methodology is based on potential and flux reconstruction [6]. In the context of mixed methods for Poisson's problems only potential reconstruction is required since an equilibrated flux is already given by the method. The proposed scheme for potential reconstruction follows three main steps: solution of the problem using an enriched space configuration for flux and potential variables (no post processing is required), smoothing of the potential variable and solution of local Dirichlet problems with hybridization. Results of some verification tests are presented illustrating the performance of the scheme.

Keywords. Mixed methods, A posteriori error estimation, Enriched FEM.

1 Introduction

In the context of finite element methods for partial differential equations, it is possible to consider a priori and a posteriori error analyses. A priori error estimates give bounds to approximation errors of the variables involved depending on regularity assumptions on the exact solution and on the approximation solution. A posteriori error estimates is based only in the approximate solution and the data of the problem, therefore, this approach is useful for efficient error control of the numerical simulations, in pratical problems where the real solution is unknown.

In this context consider the model problem

$$\nabla \cdot (-\mathbf{K}\nabla u) = f, \text{ on } \Omega, \tag{1}$$

$$u = 0, \text{ on } \partial\Omega, \tag{2}$$

defined in a polygonal region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, where $\mathbf{K}$ is a symmetric, bounded, and uniformly positive definite tensor, and $f \in L^2(\Omega)$.

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The weak form for the problem (1)-(2) can be read as,

$$(\mathbf{K}\nabla u, \nabla v)_\Omega = (f, v), \ \forall v \in H_0^1(\Omega). \tag{3}$$

Let $\mathcal{T}_h = \{K\}$ be a partition of Ω into non-overlapping elements K, such that the nonempty intersection of a distintic pair of elements is a single common node or single common edge. The term h refers to the maximum diameter h_K of the elements K.

It is known that the approximate solutions u_h for problem (3), obtained by finite element spaces based on the partition $\mathcal{T}_h$, may be continuous (e.g. for H^1-conforming variational formulations), or discontinuous (e.g. for DG, hybrid or mixed formulations). It is possible to consider discretizations whose flux approximations is not $\mathbf{H}(\text{div})$-conforming. Thus, at the approximation level, it is possible that $u_h \notin H_0^1(\Omega)$, $-\mathbf{K}\nabla u_h \notin \mathbf{H}(\text{div}, \Omega)$ or $\nabla \cdot (-\mathbf{K}\nabla u_h) \neq f$. In order to introduce a "correction" for this phenomenon, in [6] is introduced the concept of "reconstructed flux" and "reconstructed potential".

Definition 1.1. *Let u_h be the solution of the discret problem associated to (3). We will call:*

i) ***flux reconstruction****, any function $\boldsymbol{t}_h$ constructed from u_h which satisfies:*

$$\mathbf{t}_h \in \mathbf{H}(div, \Omega), \tag{4}$$

$$(\nabla \cdot \boldsymbol{t}_h, 1)_K = (f, 1)_K, \ \forall K \in \mathcal{T}_h, \tag{5}$$

ii) ***potential reconstruction****, any function s_h constructed from u_h such that*

$$s_h \in H_0^1(\Omega).$$

Using the concept of flux and potential reconstruction, in [6] is introduced the general a posterior error estimation, which is not restricted to a particular numerical method to solve the discret problem associated to (3).

Theorem 1.1. *(Theorem 7.6.1 in [6]) Let u be the weak solution of (3) and u_h any arbitrary approximation solution satisfying $u_h \in H^1(\mathcal{T}_h)^1$. Let s_h and $\boldsymbol{t}_h$ be a potential and a flux reconstruction, respectively, as in Definition 1.1. Define,*

i) $\eta_{R,K} = \dfrac{h_K}{\pi}\|f - \nabla \cdot \boldsymbol{t}_h\|_K,$

ii) $\eta_{F,K} = \|\nabla u_h + \boldsymbol{t}_h\|_K,$

iii) $\eta_{NC,K} = \|\nabla(u_h - s_h)\|_K.$

Then

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K\in\mathcal{T}_h} (\eta_{R,K} + \eta_{F,K})^2 + \sum_{K\in\mathcal{T}_h} \eta_{NC,K}^2, \tag{6}$$

where $\|\nabla v\| = \sup_{w\in H_0^1(\Omega),\|\nabla w\|=1}(\nabla v, \nabla w)$ *and π is the Poincaré constant* [2].

[1] $H^1(\mathcal{T}_h) = \{v \in L^2(\Omega_n) : v|_K \in H^1(K)\}$

[2] For all $u \in H_0^1(\Omega)$, $\|v\| \leq \dfrac{h_\Omega}{\pi}\|\nabla v\|$

Note that, if we choose a numerical scheme where the approximation solution u_h satisfy $u_h \in H_0^1(\Omega)$, for example classical Galerkin method, we can set $s_h = u_h$ and then $\eta_{NC,K} = 0$. This means that, for the a priori error estimation, we just need to find a flux reconstructed. For the other hand, if we have a numerical scheme where $-\nabla u_h \in H(\text{div})$, $(\nabla \cdot \nabla u_h, 1) = (f, 1)$ for all $K \in \mathcal{T}_h$ then defining $\boldsymbol{t}_h = -\nabla u_h$ we have $\eta_{F,K} = \eta_{R,K} = 0$, that is, for this case we just need a potential reconstruction.

2 Error estimate for the mixed method based on enriched space configurations

Given finite-dimensional approximation spaces $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$, $U_h \subset L^2(\Omega)$, consider the discrete mixed formulation for the model problem (1)-(2): Find $(\boldsymbol{\sigma}_h, u_h) \in \mathbf{V}_h \times U_h$ satisfying

$$(\boldsymbol{\sigma}_h, \mathbf{q})_\Omega - (u_h, \nabla \cdot \mathbf{q})_\Omega = 0, \ \forall \, \mathbf{q} \in \mathbf{V}_h, \tag{7}$$

$$-(\nabla \cdot \boldsymbol{\sigma}_h, v)_\Omega + (f, v)_\Omega = 0, \ \forall \, v \in U_h. \tag{8}$$

For each geometric element $K \in \mathcal{T}_h$, there is an associated master element $\hat{K}$ and an invertible geometric diffeomorfism $F_K : \hat{K} \to K$ transforming $\hat{K}$ onto K. For the present study, the elements are supposed to be affine, meaning that F_K has constant Jacobian. Vector and scalar polynomial approximations spaces $\hat{\mathbf{V}}$ and $\hat{U}$ are defined on $\hat{K}$, and are assumed to be divergence compatible, i.e., $\nabla \cdot \hat{\mathbf{V}} \equiv \hat{U}$. To compose $\mathbf{V}_h$ and U_h, the Piola transformation $\mathbb{F}_K^{\text{div}}$ or the usual mapping $\mathbb{F}_K$ (both based on F_K) are used to map $\hat{\mathbf{V}}$ and $\hat{U}$ to local divergence compatible spaces $\mathbf{V}(K)$ and $U(K)$ in the computational elements K. From (8), $\nabla \cdot \boldsymbol{\sigma}_h \in U_h$ is the L^2-orthogonal projection $\Pi_{U_h} f$ of f on U_h. Since piecewise constants are certainly included in U_h, then $-\boldsymbol{\sigma}_h \in \mathbf{H}(\text{div}, \Omega)$ is an equilibrated flux reconstruction. Using $\mathbf{t}_h = -\boldsymbol{\sigma}_h$, follows that $\eta_{F,K} = \eta_{R,K} = 0$ and as a consequence of Theorem 1.1 we obtain the following results for the a posteriori error estimate.

Theorem 2.1. *Let $u \in H_0^1(\Omega)$ be the weak solution of the model problem (3) and $\boldsymbol{\sigma} = -\mathbf{K}\nabla u \in \mathbf{H}(div, \Omega)$. Consider $\boldsymbol{\sigma}_h \in \mathbf{V}_h$ and $u_h \in U_h$ be approximate solutions given by the discrete mixed formulation (7)-(8). If $s_h \in H_0^1(\Omega)$ is a potential reconstruction, then*

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}_h} \eta_{NC,K}^2. \tag{9}$$

In order to give a fully computable a posteriori error estimate from the bound (9), we only need to specify a function $s_h \in H_0^1(\Omega)$ computed from the approximate solution. This choice is important for the precision of the estimate and is crucial to prove local efficiency.

Different ways have been proposed to reconstruct a potential $s_h \in H_0^1(\Omega)$ in order to get a fully computable a posteriori error estimate for mixed methods [1], [6], [5]. In particular, in [1] the methodology adopted is based on potential reconstrution s_h obtained by applying post-processing and averaging operators.

In this paper we propose a methodoloy consisting of three steps that are summarized as follows: solution of the problem using an enriched space configuration for flux and potential variables (no post processing is required), smoothing of the potential variable and solution of local Dirichlet problems with hybridization scheme.

Given a family of divergence compatible vector polynomial spaces $\hat{\mathbf{V}}_k$ and scalar spaces $\hat{U}_k$ defined in $\hat{K}$, suppose that a direct decomposition $\hat{\mathbf{V}}_k = \hat{\mathbf{V}}_k^{\partial} \oplus \mathring{\hat{\mathbf{V}}}_k$ holds, where $\mathring{\hat{\mathbf{V}}}_k$ indicates the flux functions with vanishing normal components over $\partial\hat{K}$. Otherwise, the functions in $\hat{\mathbf{V}}_k^{\partial}$ are assumed to have normal components over $\partial\hat{K}$ of degree k. Under these conditions, enriched versions $\hat{\mathbf{V}}_k^{n+}$ for $n \geq 1$ are defined by adding to $\hat{\mathbf{V}}_k$, higher degree internal shape functions of the original space at level $k+n$, while keeping the original border fluxes at level k, then the enriched space for flux is the denoted by $\hat{\mathbf{V}}_k^{n+} = \hat{\mathbf{V}}_k^{\partial} \oplus \mathring{\hat{\mathbf{V}}}_{k+n}$. The corresponding enriched potential spaces are now $\hat{U}_k^{n+} = \nabla \cdot \hat{\mathbf{V}}_k^{n+} = \hat{U}_{k+n}$.

For each $K \in \mathcal{T}_h$ we define the approximation spaces $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$, and $U_h \subset L^2(\Omega)$ in terms of local space configurations $\{\mathbf{V}_k^{n+}(K), U_k^{n+}(K)\}$, $n \geq 1$, backtracked from $\{\hat{\mathbf{V}}_k^{n+}, \hat{U}_k^{n+}\}$ by the transformations $\mathbb{F}_K^{\text{div}}$ and $\mathbb{F}_K$. For more details, see [2, 4].

Suppose that $\boldsymbol{\sigma}_h \in \mathbf{V}_h$ and $u_h \in U_h$ have been obtained using the enriched approximation space, so $-\boldsymbol{\sigma}_h$ is an equilibrated flux reconstruction for u_h.

Note that u_h is discontinous across element interface $e = K^l \cap K^r$, where K^l and K^r denotes the two elements of the partition $\mathcal{T}_h$ sharing the common edge e. We propose as in [1], an inter-element smoothing piecewise function $\tilde{\gamma}$ as follows.

1. For each internal face e, we define a simple average operator

$$\tilde{\gamma}|e := \frac{1}{2}\left(u_h^{K_l}|_e + u_h^{K_r}|_e\right), \tag{10}$$

   where u_h^K denotes the potential u_h restricted to element K.

2. For each node $\boldsymbol{x}_n$, define

$$\tilde{\gamma}(\boldsymbol{x}_n) := \sum_{K \in \Omega_n} \frac{u_h^K(\boldsymbol{x}_n)}{\#\Omega_n}, \tag{11}$$

   where $\Omega_n = \{K \in \mathcal{T}_h : \boldsymbol{x}_n \in K\}$ the patch of elements containing the node $\boldsymbol{x}_n$.

The resulting piecewise polinomial function has continuous trace, and can be used as a Dirichlet condition for the following local problem: Find $(\check{\boldsymbol{\sigma}}_K, \check{u}_K)$ such that

$$\check{\boldsymbol{\sigma}}_K = -\nabla \check{u}_K \text{ on } K, \tag{12}$$

$$\text{div}(\check{\boldsymbol{\sigma}}_K) = f|_K \text{ on } K, \tag{13}$$

$$\check{u}_K = \tilde{\gamma} \text{ on } \partial K. \tag{14}$$

The solution of the local problem (12) -(14) is obtained by applying a hybrid method [3] with approximations spaces $\mathbf{V}_{k+n}(K)$, $U_{k+n}(K)$ and $L_{k+n}(\partial K)$ for Lagrange multiplier space, hence the local discrete problem can be read as: Find $(\check{\boldsymbol{\sigma}}_K, \check{u}_K) \in \mathbf{V}_{k+n}(K) \times U_{k+n}(K)$ such that

$$(\check{\boldsymbol{\sigma}}_K, \mathbf{q})_K - (\check{u}_K, \nabla \cdot \mathbf{q})_K = \sum_{e \subset \partial K \setminus \partial\Omega} < \tilde{\gamma}, \mathbf{q} \cdot \boldsymbol{\eta} >_e, \ \forall \mathbf{q} \in \mathbf{V}_{k+n}(K), \tag{15}$$

$$-(\nabla \cdot \check{\boldsymbol{\sigma}}_K, v) + (f, v) = 0, \ \forall v \in U_{k+n}(K). \tag{16}$$

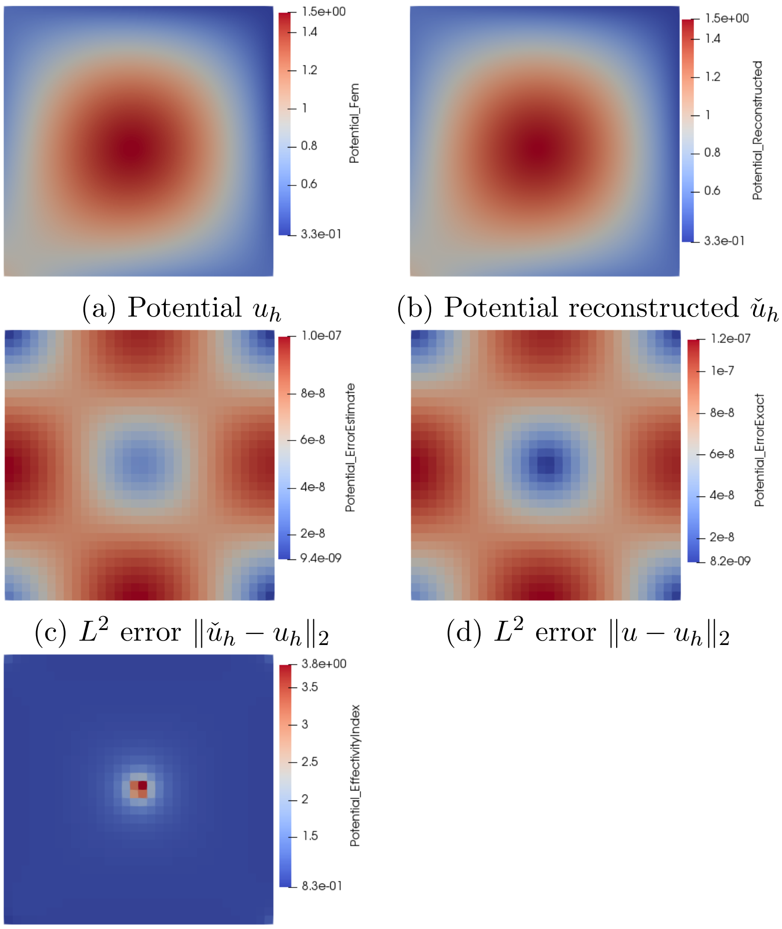
The potential reconstruction is given by $\check{u}_h = (\check{u}_K)$.

3 Numerical test

In order to illustrate the algorithm above described, we considere two different problems with $\Omega = (0,1)^2$, mesh size $h = 1/2^5$ and $k = n = 1$.

On each case the effectivity index, given by the ratio of the estimation error and true error, $I_{eff} = \dfrac{\|u_h - \check{u}_h\|}{\|u - u_h\|}$, is shown. It is known that for an optimal error estimation the effectivity index goes to one as the size mesh goes to zero, $I_{eff} \to 1$, as $h \to 0$.

Example 1: Consider a nonhomogeneous Dirichlet problem with the exact solution $u(x,y) = \sin(\pi\, x)\sin(\pi\, y) + \frac{1}{1+x+y}$. Figure 1 illustrates the potential u_h obtained by an enriched mixed method (a), the potential reconstructed (b) as the proposal described here, the L^2 estimate error (c), the L^2 exact error (d) and the I_{eff}. Note that, $I_{eff} \to 1$ except on the central elements, that can be improved with a mesh refinement.



(a) Potential u_h

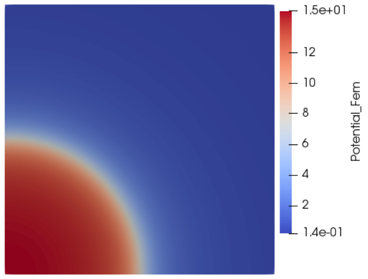
(b) Potential reconstructed $\check{u}_h$

(c) L^2 error $\|\check{u}_h - u_h\|_2$

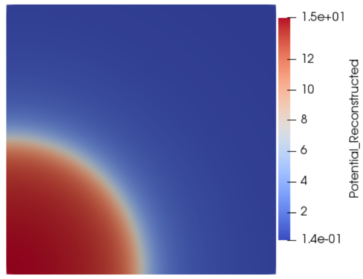
(d) L^2 error $\|u - u_h\|_2$

(e) Effectivity Index

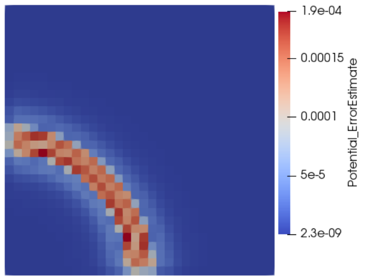
Figure 1: Non homogenous problem.



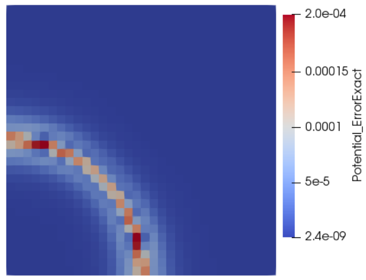
(a) Potential u_h



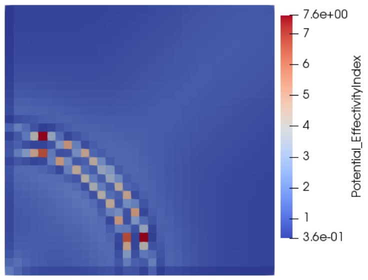
(b) Potential reconstructed $\check{u}_h$



(c) L^2 error $\|\check{u}_h - u_h\|_2$



(d) L^2 error $\|u - u_h\|_2$



(e) Effectivity Index

Figure 2: Singular problem.

Example 2: Consider a singular problem with the exact solution $u(x, y) = 5(\frac{\pi}{2} + \arctan(20 \times (0.25 - r^2))$, where $r^2 = x^2 + y^2$. Figure 2 illustrates the results using the same mesh configuration. Note that, also in this case $I_{eff} \to 1$ in most of the elements in the mesh.

4 Conclusions

We have described a method for reconstructing the potential for mixed finite element approximations. This approach does not require post processing and uses a hybrid method to solve a local problem. The numerical tests illustrated the effectiveness of the potential reconstruction indicating that it may be used as computable error estimates.

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