Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Recent progress in the research of evolution algebras associated to graphs

Paula Cadavid[1]
Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, Avenida dos Estados, 5001- Bangu - Santo André - SP, Brasil
Pablo M. Rodriguez[2]
Centro de Ciências Exatas e da Natureza, Universidade Federal de Pernambuco, Av. Prof. Moraes Rego, 1235 - Cidade Universitária - Recife - PE, Brasil

Abstract. Evolution algebras is a class of non-associative algebras with connections with many mathematical fields. One natural way to define the evolution algebra associated to a given graph is taking into account the adjacencies of the graph. In this review note we discuss recents results about these mathematical structures and we emphasize in some interesting open problems.

Keywords. Evolution Algebra, Graph, Isomorphism, Random Walk.

1 Introduction

The study of non-associative algebras is an active field of research with many applications in mathematical physics and related fields. Some well known examples of non-associative algebras are the Lie algebras and the Jordan algebras. The strong applicability of these algebras suggests that by studying new non-associative structures could lead us to discover new approaches to work in applied sciences. Our purpose is to discuss about recent results in evolution algebras, which are a special case of genetic non-associative algebras. The Theory of Evolution Algebras appeared around ten years ago in [7] as an algebraic way to mimic the self-reproduction of alleles in non-Mendelian genetics. Indeed, if one think in alleles as generators of algebras, then reproduction in genetics is represented by multiplication in the respective algebra. In the last years, different aspects of the theory of evolution algebras have seen considered and many connections with other mathematical fields, such as graph theory, stochastic processes, group theory, dynamic systems, mathematical physics, between others, have been stablished.

In this review note we focus on recents results about evolution algebras associated to graphs and we emphasize in some interesting open problems. The first definitions of evolution algebras associated to some families of graphs have been formalized by [5, 6].

---

[1]paula.cadavid@ufabc.edu.br.
[2]pablo@de.ufpe.br.

2

We refer the reader also to [4] for another look in the interplay between evolution algebras and graphs.

The rest of the paper is organized as follows. The remaining part of this section is devoted to the first definitions and to a connection between evolution algebras and discrete-time Markov chains. In Section 2 we review recent results regarding evolution algebras associated to graphs.

1.1 Evolution algebras and Markov evolution algebras

Definition 1.1. *Let $\mathcal{A} := (\mathcal{A}, \cdot)$ be an algebra over a field $\mathbb{K}$. We say that $\mathcal{A}$ is an evolution algebra if it admits a countable basis $S := \{e_i, i \in \Lambda\}$, such that*

$$\begin{aligned} e_i \cdot e_i &= \textstyle\sum_k c_{ik} e_k, \quad \text{for any } i \in \Lambda, \\ e_i \cdot e_j &= 0, \qquad\qquad \text{if } i \neq j. \end{aligned} \tag{1}$$

The scalars $c_{ik} \in \mathbb{K}$ are called the structure constants of $\mathcal{A}$ relative to S.

Note that Λ is a finite or a countable infinite index set. A basis S satisfying (1) is called a natural basis of $\mathcal{A}$. We say that $\mathcal{A}$ is real if $\mathbb{K} = \mathbb{R}$, and that it is nonnegative if it is real and the structure constants are nonnegative. In addition, if $0 \leq c_{ik} \leq 1$, and $\sum_{k=1}^{\infty} c_{ik} = 1$, for any i, k, then $\mathcal{A}$ is called a Markov evolution algebra. In this case, there is a correspondence between $\mathcal{A}$ and a discrete time Markov chain $(X_n)_{n \geq 0}$ with state space $\{x_1, x_2, \ldots, x_n, \ldots\}$ and transition probabilities given by $c_{ik} := \mathbb{P}(X_{n+1} = x_k | X_n = x_i)$, for $i, k \in \mathbb{N}^*$, and for any $n \in \mathbb{N}$, where $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. In this correspondence each state of the Markov chain is identified with a generator of S.

Remark 1.1. *The interplay between evolution algebras and Markov chains is an issue that could be explored a bit more. The best reference about this connection is Tian's book, see [7]. Chapter 4 of that book is a review of many well-known results coming from Markov chains in the context, or in the language, of Markov evolution algebras. Of course, when we have this bridge between both fields, we have a new way to describe random phenomena; i.e. through a non-associative algebras approach, which is quite interesting. However, we have to take care when dealing with these mathematical objects. Although Theorem 16 in [7, page 54] claims that for each homogeneous Markov chain, there is an evolution algebra whose structure constants are transition probabilities, and whose generator set is the state space of the Markov chain, this is not totally true. The claim is true for Markov chains with a finite state space. For the case of infinite state spaces it is not difficult to find a counter-example. Take a Branching Process $(Z_n)_{n \geq 0}$ with an offspring distribution given by a geometric random variable with parameter $p \in (0, 1)$. Notice that from any state $i \neq 0$ we can go to any state $j \in \mathbb{N}$ with positive probability. If we assume that there exists an evolution algebra whose generator set is the state space of this Markov chain, namely $S = \{e_i, i \in \mathbb{N}\}$, then by our previous remark it should be for $i \neq 0$,*

$$e_i^2 = \sum_{j \in \mathbb{N}} c_{ij} e_j, \tag{2}$$

with $c_{ij} \neq 0$ for any $j \in \mathbb{N}$. But this is a contradiction because $\mathcal{A}$ is a vector space.

1.2 Evolution algebras associated to a graph

Our purpose is to discuss about recent results on evolution algebras associated to a graph, in a sense to be defined later. So let us start with some basic notation of Graph Theory. A graph G with n vertices is a pair (V, E) where $V := \{1, \ldots, n\}$ is the set of vertices and $E := \{(i, j) \in V \times V : i \leq j\}$ is the set of edges. If $(i, j) \in E$ or $(j, i) \in E$ we say that i and j are neighbors; we denote the set of neighbors of vertex i by $\mathcal{N}(i)$ and the cardinality of this set by $\deg(i)$. We say that G is an infinite graph if it has an infinite number of vertices, i.e. V is a countable set and $|V| = \infty$. In that case we assume as an additional condition for the graph to be locally finite, i.e. $\deg(i) < \infty$ for any $i \in V$. The adjacency matrix of a given graph G, denoted by $A := A(G)$, is an $n \times n$ symmetric matrix (a_{ij}) such that $a_{ij} = 1$ if i and j are neighbors and 0, otherwise. Note that the adjacency matrix for infinite graphs is well defined. A graph is said to be singular if its adjacency matrix A is a singular matrix ($\det A = 0$), otherwise the graph is said to be non-singular. All the graphs we consider are connected, i.e. for any $i, j \in V$ there exists a positive integer n and a sequence of vertices $\gamma = (i_0, i_1, i_2, \ldots, i_n)$ such that $i_0 = i$, $i_n = j$ and $(i_k, i_{k+1}) \in E$ for all $k \in \{0, 1, \ldots, n-1\}$. For simplicity, we consider only graphs which are simple, i.e. without multiple edges or loops.

The evolution algebra induced by a graph G is defined in [7, Section 6.1] as follows.

Definition 1.2. *Let $G = (V, E)$ a graph with adjacency matrix given by $A = (a_{ij})$. The evolution algebra associated to G is the algebra $\mathcal{A}(G)$ with natural basis $S = \{e_i : i \in V\}$, and relations*

$$e_i \cdot e_i = \sum_{k \in V} a_{ik} e_k, \quad for \ i \in V, \tag{3}$$

and $e_i \cdot e_j = 0$, if $i \neq j$.

Example 1.1. *Consider a graph with 4 vertices and adjacency matrix given by*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The resulting graph, called Tadpole graph and denoted by $T_{3,1}$, is represented in Fig. 1. The evolution algebra $\mathcal{A}(T_{3,1})$ has a natural basis $S = \{e_1, e_2, e_3, e_4\}$, and relations given by $e_1^2 = e_2 + e_3 + e_4$, $e_2^2 = e_1 + e_3$, $e_3^2 = e_1 + e_2$, $e_4^2 = e_1$ and $e_i \cdot e_j = 0$, if $i \neq j$.

Another evolution algebra associated to a given graph $G = (V, E)$ is the one induced by the symmetric random walk on G, which is a discrete time Markov chain $(X_n)_{n \geq 0}$ with states space given by V and transition probabilities given by $\mathbb{P}(X_{n+1} = k | X_n = i) = a_{ik}/\deg(i)$, where $i, k \in V$, $n \in \mathbb{N}$, and $\deg(i) = \sum_{k \in V} a_{ik}$. More precisely, since the
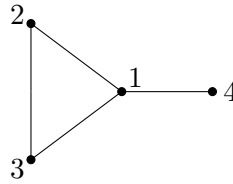
4

Figure 1: Tadpole graph $T_{3,1}$.

random walk is a discrete-time Markov chain, we can consider its related Markov evolution algebra.

Definition 1.3. *Let $G = (V, E)$ be a graph with adjacency matrix given by $A = (a_{ij})$. We define the evolution algebra associated to the SRW on G as the algebra $\mathcal{A}_{RW}(G)$ with natural basis $S = \{e_i : i \in V\}$, and relations given by*

$$e_i \cdot e_i = \sum_{k \in V} \left(\frac{a_{ik}}{\deg(i)} \right) e_k, \quad for \ i \in V, \tag{4}$$

and $e_i \cdot e_j = 0, \ if \ i \neq j$.

Example 1.2. *Consider the Tadpole graph $T_{3,1}$ of Example 1.1. In this case the evolution algebra $\mathcal{A}_{RW}(T_{3,1})$ has natural basis $S = \{e_1, e_2, e_3, e_4\}$, and relations given by $e_1^2 = \frac{1}{3}(e_2 + e_3 + e_4), \ e_2^2 = \frac{1}{2}(e_1 + e_3), \ e_3^2 = \frac{1}{2}(e_1 + e_2), \ e_4^2 = e_1$ and $e_i \cdot e_j = 0, if \ i \neq j$.*

2 Isomorphisms: recent results

One question of interest when studying evolution algebras related to graphs is, what is the connection (if any) between $\mathcal{A}_{RW}(G)$ and $\mathcal{A}(G)$ for a given graph G. This question has been an open problem, stated initially in [7,8], which has been addressed only recently in [1,3]. One way to understand such a connection is through the existence or not of isomorphisms between these algebras.

Definition 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-evolution algebras and $S = \{e_i : i \in \Lambda\}$ a natural basis for $\mathcal{A}$. We say that a non-singular $\mathbb{K}$-linear transformation $f : \mathcal{A} \longrightarrow \mathcal{B}$ is an isomorphism if $f(u) \cdot f(v) = f(u \cdot v)$, for all $u, v \in \mathcal{A}$. In this case, we denote $\mathcal{A} \cong \mathcal{B}$.*

We shall see that $\mathcal{A}_{RW}(G) \cong \mathcal{A}(G)$ provided G is well-behaved in some sense. We say that G is a d-regular graph if $\deg(i) = d$ for any $i \in V$ and some positive integer d. We say that G is a bipartite graph if its vertices can be divided into two disjoint sets, V_1 and V_2, such that every edge connects a vertex in V_1 to one in V_2. Moreover, we say that G is a biregular graph if it is a bipartite graph $G = (V_1, V_2, E)$ for which every two vertices on the same side of the given bipartition have the same degree as each other. In this case, if the degree of the vertices in V_1 is d_1 and the degree of the vertices in V_2 is d_2, then we say that G is a (d_1, d_2)-biregular graph.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics. v. 7, n. 1, 2020.

5

2.1 The case of finite non-singular graphs: a complete description

Theorem 2.1. *[3, Cadavid et al. (2018)] Let G be a finite non-singular graph. $\mathcal{A}_{RW}(G) \cong \mathcal{A}(G)$ if, and only if, G is a regular or a biregular graph. Moreover, if $\mathcal{A}_{RW}(G) \not\cong \mathcal{A}(G)$ then the only evolution homomorphism between them is the null map.*

Example 2.1. *The tadpole graph.* *The (m,n)-tadpole graph, with $m \geq 3$, consists of a cycle graph on m vertices and a path graph on n vertices, connected with a bridge. The graph considered in Example 1.1 is a $(3,1)$-tadpole graph. We adopt the usual notation $T_{m,n}$. See Fig. 2 for a general illustration of this type of graph.*
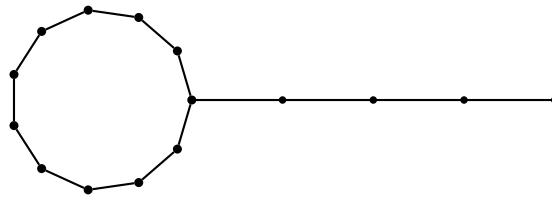

Figure 2: $(11,4)$-tadpole graph $T_{11,4}$.

First, notice that $T_{m,n}$ has $m + n$ vertices which we label as in Fig. 2. The evolution algebra $\mathcal{A}(T_{m,n})$ has a natural basis $S = \{e_1, \ldots, e_{m+n}\}$, and relations given by

$$
\begin{aligned}
&e_i^2 = e_{i-1} + e_{i+1}, &&\text{for } i \in \{1, \ldots, m-1\} \cup \{m+2, \ldots, m+n-1\}, \\
&e_m^2 = e_{m-1} + e_{m+1} + e_1, \\
&e_{n+m}^2 = e_{n+m-1}, \\
&e_i \cdot e_j = 0, &&\text{if } i \neq j.
\end{aligned}
\tag{5}
$$

On the other hand, the evolution algebra $\mathcal{A}_{RW}(T_{m,n})$ has a natural basis $S = \{e_1, \ldots, e_{m+n}\}$, and relations given by

$$
\begin{aligned}
&e_i^2 = \tfrac{1}{2}(e_{i-1} + e_{i+1}), &&\text{for } i \in \{1, \ldots, m-1\} \cup \{m+2, \ldots, m+n-1\}, \\
&e_m^2 = \tfrac{1}{3}(e_{m-1} + e_{m+1} + e_1), \\
&e_{n+m}^2 = e_{n+m-1}, \\
&e_i \cdot e_j = 0, &&\text{if } i \neq j.
\end{aligned}
\tag{6}
$$

It has been showed that $T_{m,n}$ is a non-singular graph provided m is odd. Then, we can apply Theorem 2.1 to conclude that $\mathcal{A}(T_{m,n}) \not\cong \mathcal{A}_{RW}(T_{m,n})$ as evolution algebras provided m is odd. Moreover, the only evolution homomorphism between $\mathcal{A}(T_{m,n})$ and $\mathcal{A}_{RW}(T_{m,n})$ is the null map.

6

2.2 The case of finite singular graphs: a new example and a conjecture

In order to prove Theorem 2.1 it has been important to deal with non-singular matrices, see [3] for details. Although some partial results are true also for singular matrices, see [1,3], new arguments must be developed to deal with the singular case. In this case we suggest to explore the twin partition of the graph. This has been useful to deal with other issues like the derivations space of an evolution algebra associated to a graph, see [2]. A first example in this direction is the family of trees of diameter 3 considered by [3], when we show that the only evolution homomorphism between $\mathcal{A}(G)$ and $\mathcal{A}_{RW}(G)$ is the null map. The same type of arguments, with a bit more of work, allows to obtain a similar result for a new family of graphs. We summarize this result in the following example.

Example 2.2. *The Caterpillar tree.* *The caterpillar tree is a tree in which all the vertices are within distance 1 of a central path. For the sake of simplicity in the exposition we adopt the notation $C_{a_2,a_3,...,a_{n-1}}$ for a caterpillar tree with a central path of n vertices, and such that the ith-vertex of the path has degree equals to $a_i + 2$, for $i \in \{2,...,n-1\}$ while the first and the last vertices of the path have degree equal to 1 each. See Figure 2.2 for an illustration of a caterpillar tree.*
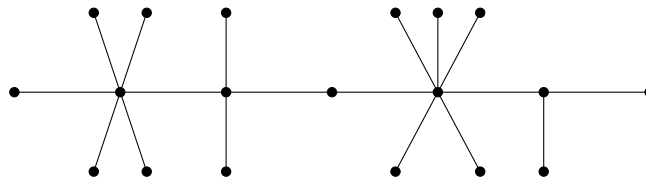

Figure 3: Caterpillar tree $C_{4,2,0,5,1}$.

It is not difficult to see that this is an example of singular path. Let $C_a := C_{a_2,a_3,...,a_{n-1}}$ be a caterpillar tree with $n + \sum_{i=2}^{n-1} a_i$ vertices, with $a_i \geq 0$ for any $i \in \{2,...,n-1\}$. Then, the only evolution homomorphism between $\mathcal{A}(C_a)$ and $\mathcal{A}_{RW}(C_a)$ is the null map. In particular, $\mathcal{A}(C_a) \ncong \mathcal{A}_{RW}(C_a)$ as evolution algebras.

Remark 2.1. *After considering the partial results from [1,3] and Example 2.2 we believe that, in fact, Theorem 2.1 should be true for singular graphs too.*

3 Conclusion

In this note we review recent results obtained by the authors related to evolution algebras associated to graphs. This is an issue of current research and its understanding could lead us to the development of new methods in applied mathematics. We suggest this because of the many applications of the subject in other mathematical fields. As we mentioned here, the problem of characterizing the connection between different evolution algebras associated to the same graph is far to be solved. The case of singular graph requires new arguments to be addressed. We believe that the twin partition of the

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics. v. 7, n. 1, 2020.

7

considered graph could have an important role in this task. On the other hand, dealing with connected graphs with an infinite number of vertices requires an approach based on results about infinite-dimensional matrices. Both problems are an interest issue for future research.

4   Acknowledgements

References

[1] P. Cadavid, M. L. Rodiño Montoya and P. M. Rodríguez. The connection between evolution algebras, random walks, and graphs, *J. Algebra Appl.*, In Press.

[2] P. Cadavid, M.L. Rodiño Montoya and P.M. Rodriguez. Characterization theorems for the space of derivations of evolution algebras associated to graphs, *Linear Multilinear Algebra*, on-line:1–15, 2018.

[3] P. Cadavid, M. L. Rodiño Montoya and P. M. Rodríguez. On the existence of isomorphisms between evolution algebras associated to graphs. eprint arXiv:1710.10516.

[4] A. Elduque and A. Labra. Evolution algebras and graphs, *J. Algebra Appl.* 14:1550103, 2015.

[5] J. Núñez, M. L. Rodríguez-Arévalo and M. T, Villar. Certain particular families of graphicable algebras, *Appl. Math. Comput.* 246:416–425, 2014.

[6] J. Núñez, M. L. Rodríguez-Arévalo and M. T, Villar. Mathematical tools for the future: Graph Theory and graphicable algebras, *Appl. Math. Comput.* 219:6113–6125, 2013.

[7] J. P. Tian. *Evolution algebras and their applications*, Springer-Verlag Berlin Heidelberg, 2008.

[8] J. P. Tian. Invitation to research of new mathematics from biology: evolution algebras, *Topics in functional analysis and algebra, Contemp. Math. 672, 257-272, Amer. Math. Soc., Providence, RI, 2016.*