Trabalho apresentado no XXXIX CNMAC, Uberlândia - MG, 2019.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Optimal Multiobjective Pulse Vaccination Campaigns in Stochastic SIR Model

Kay Adam [1]
Universidade Tecnológica de Dresden
Rodrigo T. N. Cardoso [2]
Centro Federal de Educação Tecnológica de Minas Gerais

Abstract. This article describes the results obtained with a stochastic SIR (Susceptives-Infectives-Recovered) model with impulsive vaccination campaigns. It tries to give answers to the question of how vaccination campaigns should be implemented, regarding random influences and two main targets which conflict with each other: making the vaccination campaign as cheap and handy as possible and the total number of infected persons as small as possible or the probability of eradication as greater as possible. The target of the analysis is to compare if it is better to consider the probability of eradication before or in the multiobjective optimization procedure. Results show that pre-consider the probability of eradication will be similar to the post-considering regarding their costs and quantity of infected persons, despite being computationally harder.

Keywords. Stochastic SIR, Vaccination campaigns, Multiobjective optimization.

1 Introduction

One of the most studied models describing the behavior of a disease in a population is the endemic SIR (Susceptible-Infected-Recovered) [9]. It is based on deterministic non-linear differential equations describing the influence of changes in one group on the other ones, in the same species.

This article describes the results obtained with a stochastic SIR model (SSIR), to better depict the characteristics of a disease where many little influences superpose and may lead to sudden and unpredictable changes in the behavior of the disease [1, 7],making it more realistic in small communities [3]. The stochastic model is intended to image this behavior by introducing random noise into the dynamic system. The weapon of choice is the Brownian motion since it has mean zero, it is symmetric regarding negative and positive deviations and it reflects and describes almost every situation that can be observed in nature or socio-economics.

The SSIR model is extended here to analyze the effects of impulsive vaccination on the population, and the dynamic system keeps autonomous in the time intervals between

[1] adamkay@gmx.de
[2] rodrigocardoso@cefetmg.br

DOI: 10.5540/03.2020.007.01.0414

the consecutive impulse actions of control, making the vaccination campaign as cheap and handy as possible and the total number of infected persons as small as possible or the probability of eradication as greater as possible. The idea is to give concrete suggestions to politicians or decision-makers how optimal vaccination campaigns should be implemented, given for an example a certain level of probability of eradication. The target of the analysis is to compare if it is better to consider the probability of eradication before or in the multiobjective optimization procedure [1, 7].

The article is organized as follow: Section 2 describes the SSIR model with vaccination, Section 3 presents the numerical simulation results, and Section 4 presents final discussions and the conclusions of the work.

2 The stochastic SIR model with vaccination

2.1 SSIR Model description

Let S, I, and R be the number of Susceptible, Infected, and Recovered ones, respectively. Since we want to keep the population size N constant (birth and death rate are thought of as being equal), the third variable R is always determined by the knowledge of the first two variables S and I by the relation $R(t) = N - S(t) - I(t)$ for all the time points t [10] .

As preparation for our stochastic model, we will first consider a discrete model. The corresponding stochastic model is a continuous time Markov process $\{(S_t, I_t, R_t) : t \in [0, \infty)\}$ with values in $\mathbb{Z}_+^3$. The rates from above become the conditional transition rates of the stochastic process (S, I, R). To each increment, we add and subtract its conditional expectation, conditioned on the value of the process at the beginning of the time increment of length Δt. Each increment of the process is then decomposed into the sum of the expected value of the increment and a sum of centered increments.

It is a well-known fact that for small time steps Δt and large N, a process with centered Poisson increments is well approximated by the process with Gaussian increments having the same conditional variances. Therefore we will replace the Poisson variables ΔZ_i by increments of Brownian motion or - mathematically correct - Wiener process dW_i. If we introduce dimensionless variables and use the approximation of Poisson by Gauss which leads to the use of the Wiener process, the above system becomes [3]:

$$dS = \left(\mu(N - S) - \beta\frac{SI}{N}\right) dt + G_1 dW_1 - G_2 dW_2 \tag{1}$$

$$dI = \left(\beta\frac{SI}{N} - (\gamma + \mu)I\right) dt + G_2 dW_2 - G_3 dW_3, \tag{2}$$

where

$$G_1 = \sqrt{\mu(N + S)},\ G_2 = \sqrt{\frac{\beta}{N}SI},\ G_3 = \sqrt{(\gamma + \mu)I}$$

are the according standard deviations. Therefore, for large values of N, the stochastic model will tend to the deterministic model. On the other hand, the smaller the value of N, the bigger will be the influence of the noise.

DOI: 10.5540/03.2020.007.01.0414

We implemented this model in Matlab and solved it by using the Euler-Maruyama method, the stochastic version of the Euler method for solving differential equations [8].

2.2 Behaviour of the model under influence of vaccination

The deterministic SIR model converges to a stable point which depends on the choice of the parameters. It can be shown that in the case where $\beta \geq \mu + \gamma$, the stable endemic equilibrium is given by:

$$(S^*, I^*) = N \cdot \left(\frac{\mu + \gamma}{\beta}, \frac{\mu}{\mu + \gamma} - \frac{\mu}{\beta} \right)$$

It is worth noticing that the percental equilibrium does not depend on the population size N. Note that, for extreme parameter values, the stable point can describe a situation in which the disease is almost eradicated (if $\beta << \gamma, \mu$) or in which almost the whole population gets infected (if $\beta >> \gamma, \mu$).

If we apply a vaccination campaign on the system (which means that in certain time steps a given percentage of the susceptible population gets vaccinated and becomes part of the recovered population), the formerly continuous system becomes non-continuous, with the functions S and R becoming step-size functions according to the vaccination rates.

To be precise, it means that the time in the closed interval $[0, T_f]$, being T_f the final time, has to be partitioned in a set of instants previously fixed $\Gamma = \{\tau_0, ..., \tau_M\}$, such that: $\tau_0 = 0$, $\tau_M = T_f$ and $\tau_{k+1} - \tau_k = \Delta T$ [4]. The time between each τ_k and τ_{k+1} is considered as a stage of the problem. Consider the percentage of the susceptible individuals that will be vaccinated during the campaigns: $P = \{p[0], ..., p[M-1]\}$, such that $p[k] = p(\tau_k)$, for each τ_k in Γ. Therefore, equation (1), in time τ_k, has to be updated as:

$$S(\tau_k^+) = S[k^+] = S[k](1 - p[k]); \tag{3}$$

Apart from that, it underlies the laws of the differential system from above between the vaccination time points.

2.3 Multiobjective optimization and Genetic Algorithms

In multiobjective optimization, more than one target function is considered. This relatively new approach does not give just one single solution as the result of the optimization process, but several non-dominated solutions - the so-called Pareto Front - instead. A solution is said to be non-dominated when every other solution that is better in one of the considered functions, is also worse in at least one other. A solution is only dominated by another solution if that second solution is not worse in every of the considered target functions [2, 5].

For optimizing the vaccination strategies, we consider two objective functions, related to the costs with infected and vaccinated people. We consider here two first objective functions, regarding consider the probability of eradication before or in the multiobjective optimization procedure. Therefore, firstly, the first objective function F_1 is the accumulated number of infected people during the whole time horizon, to be minimized (and the the

probability of eradication will be considered before the optimization procedure). Secondly, the first objective F_1 is the estimated probability of eradication indeed, to be maximized. The second objective function F_2 is always the cost of the campaigns, to be minimized, which depends on how many persons have to be vaccinated. The decision variables are the number of susceptible to be vaccinated in each campaign: $P = \{p[0], ..., p[M-1]\}$.

The bi-objective optimization problem is then formulated in Equations (4) and (5):

$$\text{optimize} \begin{cases} F_1 : \text{the accumulated number of infected or the probability of eradication} \\ \\ F_2 : \text{the sum of vaccined people as the total cost with the campaigns} \end{cases} \tag{4}$$

subject to:

$$\begin{cases} dS = \left(\mu(N-S) - \beta\frac{SI}{N}\right)dt + G_1 dW_1 - G_2 dW_2,\ S(0) = S_o \geq 0; \\ dI = \left(\beta\frac{SI}{N} - (\gamma+\mu)I\right)dt + G_2 dW_2 - G_3 dW_3,\ I(0) = I_o \geq 0; \\ G_1 = \sqrt{\mu(N+S)},\ G_2 = \sqrt{\frac{\beta}{N}SI},\ G_3 = \sqrt{(\gamma+\mu)I} \\ t \in (\tau_k^+, \tau_{k+1}]; \\ \\ S(\tau_k^+) = S[k^+] = S[k](1-p[k]); \\ k = 0, 1, \ldots, M-1; \\ \\ 0 < p_{min} \leq p[k] \leq p_{max} \leq 1.0; \end{cases} \tag{5}$$

The dynamic system parameters used in this work are: $\mu = 1/70$, $\beta = 0.98$, $\gamma = 1/24$, $N = 200$, and each SSIR is simulated in $1,500$ trial runs. The initial condition is $(s^*, i^*, r^*) = (0.06, 0.24, 0.70)$, We consider $T_f = 360$ $u.t.$, $p_{min} = 0.4$, $p_{max} = 0.8$.

The multiobjective optimization model offers a whole set of optimal solutions regarding the total number of infected people as well as the costs of the campaigns. Decision-makers can then decide for one of these optimal campaigns, according to other variables like budget, media influence, society's view of the disease, guidelines by law or politics and so on.

Genetic algorithms intend to copy the behavior of nature who optimizes using the principles of selection, mutation, and recombination. Here a set of solutions is considered. The set of these solutions undergoes a selection process, making up a set of parents with good genes for the next rounds. From this set of considerably good solutions, a new population is created by employing the principles of mutation and recombination. This new set then undergoes the selection process again. Solutions are always compared regarding their state of being dominated by others or being dominant for other solutions. At the end of the process the best solutions, those who are not dominated by any other solution, form the Pareto Front. One of the most famous multiobjective algorithm, which is used in this work, is the NSGA II [6].

A parallel version of the standard NSGA-II is considered here, distributing the evaluation of the objective function. Each algorithm is tested in 30 independent executions in 8 cores of a standard workspace. The population size is set to 40, the number of generations is 100, the probability of crossover is 0.8, and the probability of mutation is 0.05.

DOI: 10.5540/03.2020.007.01.0414

3 Results

In Section 3.1, the probability of eradication (POE) is post-considered, to decide which nondominated vaccination police must be implemented. The first objective F_1 is considered as the accumulated number of infected people during the time horizon. In Section 3.2, the first objective F_1 is the estimated probability of eradication (POE) indeed. The second objective function F_2 is set to be the cost of the campaigns in the optimization procedure of Sections 3.1 and 3.2.

3.1 Post-considering of POE

In this section, we use the NSGA-II for optimizing the vaccination campaigns considering the deterministic counterpart of SSIR, using F_1 set to minimize the accumulated number of infected people.

We compute the POE for each nondominated campaign come from the optimization process, Figure 3.1 shows the campaigns of the Pareto Front with the respective POE. Note that the rightmost campaigns within the Pareto Front have POE equal to zero, according to the fact that their integral of infected persons is highest.

Since all the rightmost campaigns show to have a POE equal to zero, we can replace them all by the leftmost, since choosing a campaign more to the right would bring no further advantage, except for smaller costs. The same holds for the leftmost campaigns from the Pareto Front: since all of them share a POE equal to one, they could be reduced to the rightmost of these campaigns, since choosing a campaign more to the left would also eradicate the disease, with fewer people getting infected until that point. In this way, note that we can discard some of the campaigns proposed by the NSGA-II, and the Pareto Front can, therefore, be reduced to a few points in the middle.

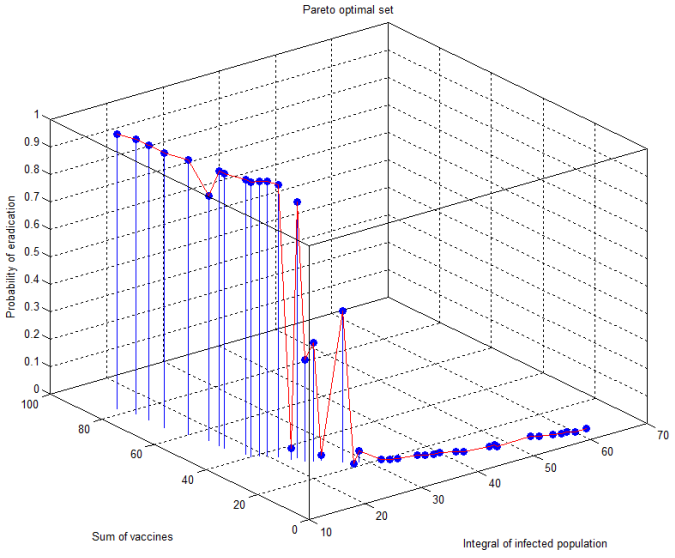

Figura 1: POE for each policy in the Pareto Front.

DOI: 10.5540/03.2020.007.01.0414

3.2 Pre-considering of POE

In this section, we use the NSGA-II for optimizing the vaccination campaigns considering the SSIR, using F_1 set to maximize the probability of eradication. We replace the first objective function - the integral of infected people - by the POE of the vaccination strategy and optimize using the NSGA-II.

The Pareto Front of the optimization process looks like in the Figure 3.2. If we have a closer look at how different strategies influence the population sizes of susceptibles and infectives, we can note that a strategy which does not lead to the theoretical extinction of the disease is similar to one who does not, in relation to the ratio of susceptibles to infecteds over time. If we compute the integral of infected persons for those campaigns, we see that the few newly found campaigns resemble very much the old ones which were left after post-considering the POE, discarding some solutions from the first Pareto Front.

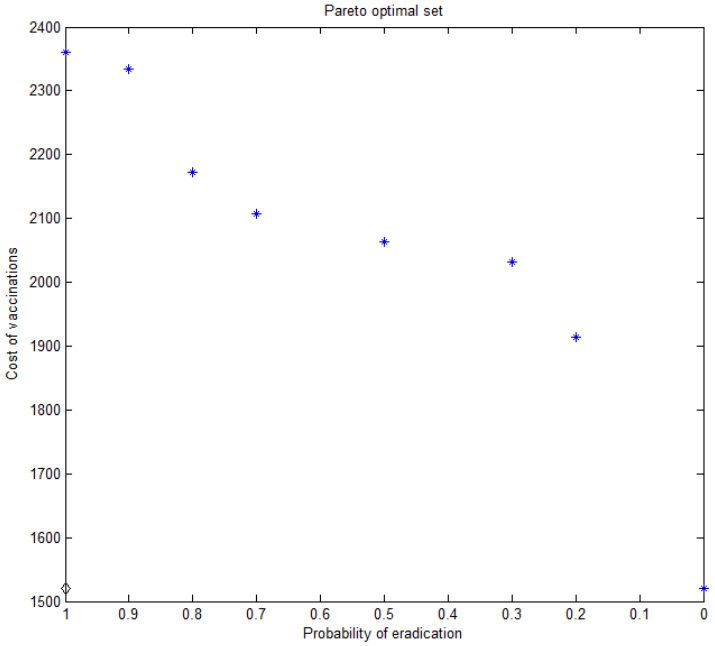

Figura 2: Pareto Front considering the POE as objective function.

4 Conclusions

This work studies how to take into account the POE for a given campaign in the multiobjective optimization procedure using SSIR. We show that by taking into account the POE, the number of campaigns under examination may be reduced significantly. The complete population may be broken down into smaller partitions depending on the vaccination factor that is used for defending against the disease. If we consider the POE from the beginning and use it as an objective function inside the optimization procedure,

DOI: 10.5540/03.2020.007.01.0414

similar results can be achieved, which means that the proposed strategies are similar, regarding their costs and their quantity of infected persons and the relation to the ratio of susceptibles to infecteds over time.

Acknowledgements

The authors would thank to the Brazilian Agencies CNPq, FAPEMIG and to CEFET-MG and IAESTE for financial support.

5 References

Referências

[1] L. J. S. Allen. An Introduction to Stochastic Epidemic Models. *An Introduction to Mathematical Biology*. Prentice Hall, 2006.

[2] T. Bäck. *Evolutionary Algorithms in Theory and Practice: Evolution Strategies, Evolutionary Programming, Genetic Algorithms*. Oxford Univ. Press., 1996.

[3] T. Britton. Stochastic epidemic models: a survey. *Mathematical Biosciences*, volume 225, Elsevier, 2010.

[4] A. R. Cruz, R.T.N. Cardoso, R.H.C. Takahashi. Multiobjective synthesis of robust vaccination policies. *Applied Soft Computing*, 50: 34-47, 2017.

[5] K. Deb. *Multi-Objective Optimization Using Evolutionary Algorithms*, John Wiley and Sons, 2001.

[6] K. Deb, S. Agrawal, A. Pratap. A fast elitist non-dominated sorting genetic algorithm for multi-objective optimization: NSGA-II, *International Conference on Parallel Problem Solving from Nature (PPSN)*. Springer, Berlin, Heidelberg, 2000.

[7] L.C. Evans. *An Introduction to Stochastic Differential Equations*, Department of Mathematics, University of Berkeley, 2013.

[8] D.J. Higham. An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations, *SIAM Review*, volume 43, 2001.

[9] W. Kermack and A. McKendrick. A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society of London Series A*, 15:700-721, 1927.

[10] R. Kuske, L.F.Gordillo, P. Greenwood. Sustained oscillations via coherence resonance in SIR. *Journal of Theoretical Biology*, 245:459-469, 2007.

