Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 2, N. 1, 2014.
Trabalho apresentado no CMAC-Sul, Curitiba-PR, 2014.

A wavelet Galerkin approximation of Fredholm integral eigenvalue problems with bidimensional Haar functions

Saulo P. Oliveira, **Felipe Wisniewski**,
Depto de Matemática, UFPR,
81531-980, Curitiba, PR
E-mail: saulopo@ufpr.br, felipewisniewski@yahoo.com.br,

Juarez S. Azevedo
Centro de Ciências Exatas e Tecnológicas, UFRB
44380-000, Cruz das Almas, BA
E-mail: juarezsa@gmail.com.

Resumo: *We consider the numerical approximation of homogeneous Fredholm integral equations of second kind. We employ the wavelet Galerkin method with 2D Haar wavelets as shape functions. We thoroughly describe the derivation of the shape functions and present a preliminary numerical experiment illustrating the computation of eigenvalues for a particular covariance kernel.*

Palavras-chave: *Fredholm integral equations, Galerkin method, 2D Haar wavelets*

1 Introduction

A large class of random processes, stationary or non-stationary, can be expressed in terms of deterministic orthogonal functions and uncorrelated random variables. This representation is obtained from the eigenvalues and eigenfunctions of a homogeneous Fredholm integral equation of the second kind whose kernel is given by the covariance function of the random process. This approach has been widely used in the parametrization of parameters for elasticity problems, heat and mass transfer, fluid mechanic and acoustic [1, 6, 9].

In order to obtain a high performance in the representation of random processes it is necessary to accurately calculate the eigenpairs of the covariance function. In some special cases, these eigenpairs can be analytically obtained [9], but in general one needs to resort to the numerical discretization of Fredholm integral equation. Several numerical methods have been used with this purpose (see [7] and the references therein).

In this study, we highlight the Galerkin method with wavelet shape functions [3, 4, 5, 8] for the computation of the eigenpairs of the covariance in the two-dimensional case. To accomplish the task, we use two-dimensional Haar wavelet basis functions [2]. A similar study was conducted in [8] and treated only the one-dimensional case.

The work is organized in the following way: in Section 2, two-dimensional Haar wavelets are introduced. In Section 3, the formulation of the method based on Haar wavelets and the discretization of the linear two dimensional integral equation of second kind is described. In Section 4 a preliminary example illustrates the proposed method.

2 Problem setting

Let $D = [0, 1] \times [0, 1]$. We consider the Hilbert space $L^2(D)$ of real-valued functions equipped with the usual inner product and induced norm

$$\langle u, v \rangle = \int_D u(\boldsymbol{x}) v(\boldsymbol{x})\, d\boldsymbol{x}, \quad \|v\|_{L^2(D)} = \langle v, v \rangle^{1/2}. \tag{1}$$

Let $K : D \times D \to I\!R$ be be a symmetric covariance kernel. We assume that $K(\boldsymbol{x}, \boldsymbol{y})$ is of nonnegative-definite type, i.e., for any finite subset $D_n \subset D$ and for any function $u : D_n \to I\!R$,

$$\sum_{\boldsymbol{x},\boldsymbol{y} \in D_n} K(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{x}) u(\boldsymbol{y}) \geq 0. \tag{2}$$

We have that K admits the spectral decomposition

$$K(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{\infty} \lambda_k u_k(\boldsymbol{x}) u_k(\boldsymbol{y}), \tag{3}$$

where the nonnegative eigenvalues λ_k and the orthonormal eigenfunctions u_k ($k \geq 1$) are the solutions of the homogeneous Fredholm integral equation

$$\int_D K(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y})\, d\boldsymbol{y} = \lambda u(\boldsymbol{x}), \quad \boldsymbol{x} \in D. \tag{4}$$

2.1 Variational formulation and Galerkin approximation

Let us consider the variational formulation of the Fredholm integral equation (4): find $\lambda_k \in \mathbb{R}$ and $u_k(\boldsymbol{x}) \in L^2(D)$ ($k = 1, 2, \ldots$) such that

$$a(u_k, v) = \lambda_k \langle u_k, v \rangle \quad \forall\, v \in L^2(D), \tag{5}$$

$$a(u, v) = \int_D \int_D K(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y}) v(\boldsymbol{x})\, d\boldsymbol{y}\, d\boldsymbol{x}. \tag{6}$$

Let V_h be a finite-dimensional subspace of $L^2(D)$ and let $\{v_1, \ldots, v_N\} \subset L^2(D)$ be a basis of V_h. The Galerkin approximation to (5) in V_h consists of finding $\lambda_k^h \in I\!R$ and $u_k^h(\boldsymbol{x}) \in V_h$ ($1 \leq k \leq N$) such that

$$a(u_k^h, v_h) = \lambda_k^h \langle u_k^h, v_h \rangle \quad \forall\, v_h \in V_h. \tag{7}$$

Since $\{v_1, \ldots, v_N\}$ is a basis of V_h, we have that (7) is equivalent to solving the system of equations

$$a(u_k^h, v_i) = \lambda_k^h \langle u_k^h, v_i \rangle \quad 1 \leq i \leq N. \tag{8}$$

Let us write u_k^h with respect to the basis of V_h:

$$u_k^h(\boldsymbol{x}) = \sum_{j=1}^{N} u_j v_i(x, y), \tag{9}$$

By substituting (9) into (8) and using the linearity $a(\cdot, \cdot)$, we find

$$\sum_{j=1}^{N} u_j a(v_j, v_i) \;=\; \lambda_k^h \sum_{j=1}^{N} u_j \langle v_j, v_i \rangle \quad 1 \leq i \leq N. \tag{10}$$

In matrix form, we have the generalized eigenvalue problem $K\boldsymbol{u}_k = \lambda_k^h W \boldsymbol{u}_k$, where the matrices K and W are defined by the coefficients

$$K_{i,j} = a(v_j, v_i), \quad W_{i,j} = \langle v_j, v_i \rangle, \quad 1 \leq i, j \leq N. \tag{11}$$

If the basis functions $v_1, \ldots, v_N$ are orthonormal, then $W = I$ and the problem (10) reduces to the standard eigenvalue problem $K\boldsymbol{u}_k = \lambda_k^h \boldsymbol{u}_k$.

DOI: 10.5540/03.2014.002.01.0060

3 Selection of the basis functions

We build the shape functions $v_i,\ 1 \leq i \leq N$, from the 1D Haar scaling function $\phi(t) = \chi_{[0,1)}(t)$ and the 1D Haar mother wavelet $\psi(t) = \chi_{[0,1/2)}(t) - \chi_{[1/2,1/1)}(t)$, i.e.,

$$\phi(t) = \begin{cases} 1 & 0 \leq t < 1, \\ 0 & \text{otherwise.} \end{cases}, \quad \psi(t) = \begin{cases} 1 & 0 \leq t < 1/2, \\ -1 & 1/2 \leq t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

From ϕ and ψ we define the functions $\phi_{m,n}(x) = 2^{m/2}\phi(2^m x - n)$ and $\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)$, and then, following [2], the 2D functions

$$\begin{cases} \psi^{(1)}_{m,n,l}(x,y) = \phi_{m,n}(x)\psi_{m,l}(y), \\ \psi^{(2)}_{m,n,l}(x,y) = \psi_{m,n}(x)\phi_{m,l}(y), \\ \psi^{(3)}_{m,n,l}(x,y) = \psi_{m,n}(x)\psi_{m,l}(y). \end{cases} \tag{13}$$

For simplicity let us initially partition the domain D into $2^M \times 2^M$ square elements. Our first basis function is

$$v_1(x,y) = \phi_{0,0}(x)\phi_{0,0}(y), \tag{14}$$

whereas for $2 \leq i \leq 2^{2M}$ we define the shape functions v_i as

$$v_i(x,y) = \psi^{(k)}_{m,n,l}(x,y), \tag{15}$$

where the indices $0 \leq m \leq M-1$, $0 \leq n, l \leq 2^m - 1$, and $k = 1,2,3$ are related to the global index i as follows:

$$i = 2^{2m} + 3(2^m\, l + n) + k \quad (i > 1). \tag{16}$$

Let us also notice that

$$\phi_{m,n}(x) = \frac{1}{\sqrt{2}}\left[\phi_{m+1,2n}(x) + \phi_{m+1,2n+1}(x)\right], \tag{17}$$

$$\psi_{m,n}(x) = \frac{1}{\sqrt{2}}\left[\phi_{m+1,2n}(x) - \phi_{m+1,2n+1}(x)\right]. \tag{18}$$

From (13), (24), (17) and (18), we can rewrite the 2D functions as follows:

$$\begin{aligned} \psi^{(0)}_{m,n,l}(x,y) &= \frac{1}{2}\Big[\psi^{(0)}_{m+1,2n,2l}(x,y) + \psi^{(0)}_{m+1,2n,2l+1}(x,y) \\ &\quad + \psi^{(0)}_{m+1,2n+1,2l}(x,y) + \psi^{(0)}_{m+1,2n+1,2l+1}(x,y)\Big], \end{aligned} \tag{19}$$

$$\begin{aligned} \psi^{(1)}_{m,n,l}(x,y) &= \frac{1}{2}\Big[\psi^{(0)}_{m+1,2n,2l}(x,y) - \psi^{(0)}_{m+1,2n,2l+1}(x,y) \\ &\quad + \psi^{(0)}_{m+1,2n+1,2l}(x,y) - \psi^{(0)}_{m+1,2n+1,2l+1}(x,y)\Big], \end{aligned} \tag{20}$$

$$\begin{aligned} \psi^{(2)}_{m,n,l}(x,y) &= \frac{1}{2}\Big[\psi^{(0)}_{m+1,2n,2l}(x,y) + \psi^{(0)}_{m+1,2n,2l+1}(x,y) \\ &\quad - \psi^{(0)}_{m+1,2n+1,2l}(x,y) - \psi^{(0)}_{m+1,2n+1,2l+1}(x,y)\Big], \end{aligned} \tag{21}$$

$$\begin{aligned} \psi^{(3)}_{m,n,l}(x,y) &= \frac{1}{2}\Big[\psi^{(0)}_{m+1,2n,2l}(x,y) - \psi^{(0)}_{m+1,2n,2l+1}(x,y) \\ &\quad - \psi^{(0)}_{m+1,2n+1,2l}(x,y) + \psi^{(0)}_{m+1,2n+1,2l+1}(x,y)\Big]. \end{aligned} \tag{22}$$

DOI: 10.5540/03.2014.002.01.0060

3.1 Assembling the discrete eigenvalue system

In order to compute the entries $K_{i,j} = a(v_j, v_i)$ of the discrete eigenvalue system, let us introduce the auxiliary coefficients

$$\begin{aligned}
\tilde{K}^{(0)}_{n_i,l_i,n_j,l_j} &= a(\psi^{(0)}_{M,n_j,l_j}, \psi^{(0)}_{M,n_i,l_i}),\\
\tilde{K}^{(1)}_{m_i,n_i,l_i,m_j,n_j,l_j} &= a(\psi^{(0)}_{m_j,n_j,l_j}, \psi^{(0)}_{m_i,n_i,l_i}),\\
\tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j,n_j,l_j} &= a(\psi^{(0)}_{m_j,n_j,l_j}, \psi^{(k_i)}_{m_i,n_i,l_i}),\\
\tilde{K}^{(3)}_{m_i,n_i,l_i,k_j,m_j,n_j,l_j} &= a(\psi^{(k_j)}_{m_j,n_j,l_j}, \psi^{(0)}_{m_i,n_i,l_i}),\\
\tilde{K}^{(4)}_{k_i,m_i,n_i,l_i,k_j,m_j,n_j,l_j} &= a(\psi^{(k_j)}_{m_j,n_j,l_j}, \psi^{(k_i)}_{m_i,n_i,l_i}),
\end{aligned} \tag{23}$$

and the additional function

$$\psi^{(0)}_{m,n,l}(x,y) = \phi_{m,n}(x)\phi_{m,l}(y). \tag{24}$$

In analogy with (16), the arrays above may be arranged as matrices as follows:

$$\begin{aligned}
\tilde{K}^{(0)}_{n_i,l_i,n_j,l_j} &= \tilde{K}^{(0)}_{2^M l_i+n_i+1,2^M l_j+n_j+1},\\
\tilde{K}^{(1)}_{m_i,n_i,l_i,m_j,n_j,l_j} &= \tilde{K}^{(1)}_{(2^{2m_i}-1)/3+2^{m_i}l_i+n_i+1,(2^{2m_j}-1)/3+2^{m_j}l_j+n_j+1},\\
\tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j,n_j,l_j} &= \tilde{K}^{(2)}_{2^{2m_i}+3(2^{m_i}\,l_i+n_i)+k_i,(2^{2m_j}-1)/3+2^{m_j}l_j+n_j+1},\\
\tilde{K}^{(3)}_{m_i,n_i,l_i,k_j,m_j,n_j,l_j} &= \tilde{K}^{(3)}_{(2^{2m_i}-1)/3+2^{m_i}l_i+n_i+1,2^{2m_j}+3(2^{m_j}\,l_j+n_j)+k_j},\\
\tilde{K}^{(4)}_{k_i,m_i,n_i,l_i,k_j,m_j,n_j,l_j} &= \tilde{K}^{(4)}_{2^{2m_i}+3(2^{m_i}\,l_i+n_i)+k_i,2^{2m_j}+3(2^{m_j}\,l_j+n_j)+k_j}.
\end{aligned} \tag{25}$$

Furthermore, note from (23) that

$$K_{i,j} = \begin{cases}
\tilde{K}^{(1)}_{0,0,0,0,0,0}, & i=j=1,\\
\tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,0,0,0}, & 2 \le i \le n,\ j=1,\\
\tilde{K}^{(3)}_{0,0,0,k_j,m_j,n_j,l_j}, & i=1,\ 2 \le j \le n,\\
\tilde{K}^{(4)}_{k_i,m_i,n_i,l_i,k_j,m_j,n_j,l_j}, & 2 \le i,j \le n.
\end{cases} \tag{26}$$

In order to calculate the coefficients $\tilde{K}^{(0)}_{n_i,l_i,n_j,l_j}$, $(0 \le n_i, l_i, n_j, l_j < 2^M)$ we use relations (5) and (9) along with the rectangle quadrature rule:

$$\tilde{K}^{(0)}_{n_i,l_i,n_j,l_j} = 2^{-2M} K(\overline{x}_{M,n_j}, \overline{x}_{M,l_j}, \overline{x}_{M,n_i}, \overline{x}_{M,l_i}), \quad \overline{x}_{M,n} = 2^{-M}\left(n + \frac{1}{2}\right).$$

Note that $\tilde{K}^{(0)}_{n_i,l_i,n_j,l_j}$ is, up to index permutations, the matrix of the piecewise-constant finite element method for (4) [1]. In the following we describe a pyramid algorithm [2] to recover $\tilde{K}^{(4)}_{k_i,m_i,n_i,l_i,k_j,m_j,n_j,l_j}$, avoiding the need of further integrations.

We use relations (19)-(22) and the bilinearity of $a(\cdot,\cdot)$ to compute the coefficients $\tilde{K}^{(1)}_{m_i,n_i,l_i,M,n_j,l_j}$ for $0 \le n_i, l_i < 2^{m_i}$, as well as the coefficients $\tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,M,n_j,l_j}$ for $0 \le n_j, l_j < 2^M$.

For instance, we have from (19) and (23) that, for $0 \le n_i, l_i < 2^{M-1}$,

$$\tilde{K}^{(1)}_{M-1,n_i,l_i,M,n_j,l_j} = \frac{1}{2}\left[\tilde{K}^{(0)}_{2n_i,2l_i,n_j,l_j} + \tilde{K}^{(0)}_{2n_i,2l_i+1,n_j,l_j} + \tilde{K}^{(0)}_{2n_i+1,2l_i,n_j,l_j} + \tilde{K}^{(0)}_{2n_i+1,2l_i+1,n_j,l_j}\right].$$

DOI: 10.5540/03.2014.002.01.0060

In general, we have for $m_i < M$ and $0 \leq n_i, l_i < 2^{m_i}$ that

$$\begin{aligned}
\tilde{K}^{(1)}_{m_i,n_i,l_i,M,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(1)}_{m_i+1,2n_i,2l_i,M,n_j,l_j} + \tilde{K}^{(1)}_{m_i+1,2n_i,2l_i+1,M,n_j,l_j} \\
&\quad + \tilde{K}^{(1)}_{m_i+1,2n_i+1,2l_i,M,n_j,l_j} + \tilde{K}^{(1)}_{m_i+1,2n_i+1,2l_i+1,M,n_j,l_j}\Big], \\
\tilde{K}^{(2)}_{1,m_i,n_i,l_i,M,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(1)}_{m_i+1,2n_i,2l_i,M,n_j,l_j} - \tilde{K}^{(1)}_{m_i+1,2n_i,2l_i+1,M,n_j,l_j} \\
&\quad + \tilde{K}^{(1)}_{m_i+1,2n_i+1,2l_i,M,n_j,l_j} - \tilde{K}^{(1)}_{m_i+1,2n_i+1,2l_i+1,M,n_j,l_j}\Big], \\
\tilde{K}^{(2)}_{2,m_i,n_i,l_i,M,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(1)}_{m_i+1,2n_i,2l_i,M,n_j,l_j} + \tilde{K}^{(1)}_{m_i+1,2n_i,2l_i+1,M,n_j,l_j} \\
&\quad - \tilde{K}^{(1)}_{m_i+1,2n_i+1,2l_i,M,n_j,l_j} - \tilde{K}^{(1)}_{m_i+1,2n_i+1,2l_i+1,M,n_j,l_j}\Big], \\
\tilde{K}^{(2)}_{3,m_i,n_i,l_i,M,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(1)}_{m_i+1,2n_i,2l_i,M,n_j,l_j} - \tilde{K}^{(1)}_{m_i+1,2n_i,2l_i+1,M,n_j,l_j} \\
&\quad - \tilde{K}^{(1)}_{m_i+1,2n_i+1,2l_i,M,n_j,l_j} + \tilde{K}^{(1)}_{m_i+1,2n_i+1,2l_i+1,M,n_j,l_j}\Big].
\end{aligned}$$

Afterwards, we compute $\tilde{K}^{(4)}$ from $\tilde{K}^{(2)}$ following the same procedure:

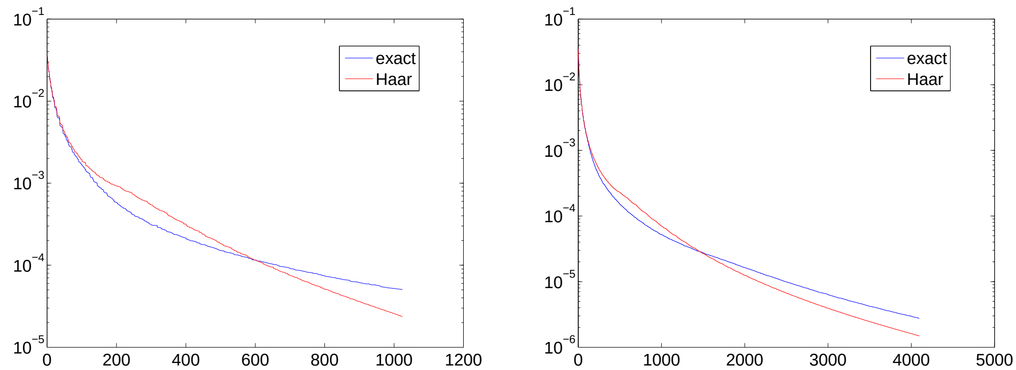
$$\begin{aligned}
\tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j,2l_j} + \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j,2l_j+1} \\
&\quad + \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j+1,2l_j} + \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j+1,2l_j+1}\Big], \\
\tilde{K}^{(4)}_{k_i,m_i,n_i,l_i,1,m_j,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j,2l_j} - \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j,2l_j+1} \\
&\quad + \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j+1,2l_j} - \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j+1,2l_j+1}\Big], \\
\tilde{K}^{(4)}_{k_i,m_i,n_i,l_i,2,m_j,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j,2l_j} + \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j,2l_j+1} \\
&\quad - \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j+1,2l_j} - \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j+1,2l_j+1}\Big], \\
\tilde{K}^{(4)}_{k_i,m_i,n_i,l_i,3,m_j,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j,2l_j} - \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j,2l_j+1} \\
&\quad - \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j+1,2l_j} + \tilde{K}^{(2)}_{k_i,m_i,n_i,l_i,m_j+1,2n_j+1,2l_j+1}\Big].
\end{aligned}$$

We also need the entries of $\tilde{K}^{(3)}_{0,0,0,k_j,m_j,n_j,l_j}$ in (26):

$$\begin{aligned}
\tilde{K}^{(3)}_{0,0,0,1,m_j,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j,2l_j} - \tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j,2l_j+1} \\
&\quad + \tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j+1,2l_j} - \tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j+1,2l_j+1}\Big], \\
\tilde{K}^{(2)}_{0,0,0,2,m_j,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j,2l_j} + \tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j,2l_j+1} \\
&\quad - \tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j+1,2l_j} - \tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j+1,2l_j+1}\Big], \\
\tilde{K}^{(2)}_{0,0,0,3,m_j,n_j,l_j} &= \frac{1}{2}\Big[\tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j,2l_j} - \tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j,2l_j+1} \\
&\quad - \tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j+1,2l_j} + \tilde{K}^{(1)}_{0,0,0,m_j+1,2n_j+1,2l_j+1}\Big].
\end{aligned}$$

4 Numerical results

In this section, we applied the method presented in this work to calculate the eigenvalues of separable exponential covariance function $K(\mathbf{x}, \mathbf{y}) = \exp(-|x_1 - y_1|/\eta - |x_2 - y_2|/\eta)$ ($\eta = 0.1$), whose exact eigenvalues and eigenfunctions are known [9]. Numerical results obtained using Matlab are shown and compared with the exact eigenvalues in Figure 1.



(a) For M = 5 (b) For M = 6

Figure 1: Eigenvalues generated by the method of 2D Haar Wavelets and exact eigenvalues

5 Final remarks

We have extended the work by Phoon et al [8] to a two-dimensional domain, although (non-homogeneous) Fredholm integral equations have been solved with 2D Haar wavelets elsewhere [4]. Forthcoming work comprises testing additional covariance functions and providing error estimates for the approximate eigenvalues and eigenfunctions.

References

[1] J. S. Azevedo, M. A. Murad, M. R. Borges, and S. P. Oliveira. A space–time multiscale method for computing statistical moments in strongly heterogeneous poroelastic media of evolving scales. *Int. J. Numer. Meth. Engrg.*, 90(6):671–706, 2012.

[2] G. Beylkin, R. Coifman, and V. Rokhlin. Fast wavelet transforms and numerical algorithms. I. *Commun. Pure Appl. Math.*, 44(2):141–183, 1991.

[3] J. Castrillon-Candas and K. Amaratunga. Fast Estimation of Continuous Karhunen-Loève Eigenfunctions Using Wavelets. *IEEE Trans. Signal Process.*, 50:205–228, 2002.

[4] H.-A. Derili, S. Sohrabi, and A. Arzhang. Two-dimensional wavelets for numerical solution of integral equations. *Math. Sci.*, 6:4, 2012.

[5] C. A. Micchelli, Y. Xu, and Y. Zhao. Wavelet Galerkin methods for second-kind integral equations. *J. Comput. Appl. Math.*, 86(1):251–270, 1997.

[6] M. Motamed, F. Nobile, and R. Tempone. A stochastic collocation method for the second order wave equation with a discontinuous random speed. *Numerische Mathematik*, 123(3):493–536, 2013.

[7] S. P. Oliveira and J. S. Azevedo. Spectral element approximation of Fredholm integral eigenvalue problems. *J. Comput. Appl. Math.*, 257:46–56, 2014.

[8] K. K. Phoon, S. P. Huang, and S. T. Quek. Implementation of Karhunen-Loève expansion for simulation using a wavelet-Galerkin scheme. *Prob. Eng. Mech.*, 17(3):293–303, 2002.

[9] D. Zhang and Z. Lu. An efficient, high-order pertubation approach for flow in random porous media via Karhunen-Loève and polynomial expansions. *J. Comput. Phys.*, 194(2):773–794, 2004.

