Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 2, N. 1, 2014.
Trabalho apresentado no CMAC-Sul, Curitiba-PR, 2014.

Computing Directional Galperin's Rates

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Abstract: *Elements of the d-dimensional real space $\mathbb{R}^d$ are called points. Maps $x : \mathbb{R}^d \to M$ are called configurations, where $M = \{0, 1, 2\}$. Any configuration x is determined by its components x_p for all points $p \in \mathbb{R}^d$. The configuration all of whose components are zeros is called "all zeros". Two configurations x and y are called close to each other if the set $\{p \in \mathbb{R}^d : x_p \neq y_p\}$ is bounded. A configuration is called an island if it is close to "all zeros". The set of configurations is denoted by $\Omega = M^{\mathbb{R}^d}$. Any map from Ω to Ω is called an operator. We say that an operator D erodes an island x if there is a natural t such that xD^t (the result of t iterative applications of D to x) is "all zeros". We call an operator D an eroder if it erodes all islands. Galperin have obtained an eroder criteria for one-dimensional cellular automata by using his left and right rates [1]. Galperin found a way of computing his rates [2] in the discrete space, but he presented no detailed routine or implementation for computing them. Later de Santana generalized these rates introducing the directinal Galperin's rates. Directional Galperin's rates were employed in studying erodicity of two-dimensional cellular automata [5]. Here we present an algoritm for computing Galperin's rates. Furthermore, from this algorithm we can compute some directional Galperin's rates.*

Keywords: *Cellular Automata, Monotonicity, Eroder, Directional Galperin Rates.*

Let d be a natural number. The d-dimensional real space $\mathbb{R}^d$ is called *Space* and its elements are called *points*. Elements of the set $M = \{0, 1, 2\}$ are called *states* and maps $x : \mathbb{R}^d \to M$ are called *configurations*. Any configuration x is determined by its *components* $x_p \in M$ for all points $p \in \mathbb{R}^d$. The configuration all of whose components are zeros is called "all zeros". A configuration is called an *island* if it is close to "all zeros".

The set of configurations is denoted by $\Omega_d = M^{\mathbb{R}^d}$. Any map from Ω_d to Ω_d is called an *operator*. Throughout this article we write operators on the right side of configurations on which they act. We call a configuration x *invariant* for an operator D if $xD = x$.

We order M in the evident way $0 < 1 < 2$ and introduce a partial order on Ω as follows: given $x, y \in \Omega_d$, we say that x *preceeds* y and write $x \prec y$ if $x_p \leq y_p$ for all points $p \in \mathbb{R}^d$. We call an operator $D : \Omega_d \to \Omega_d$ *monotonic* if

$$\forall\, x, y \in \Omega_d : x \prec y \Longrightarrow xD \prec yD.$$

For any point $p \in \mathbb{R}^d$ we define a *shift* $\text{Shift}^p : \Omega_d \to \Omega_d$ by the rule

$$(x\ \text{Shift}^p)_q = x_{q-p} \quad \text{for all} \quad q \in \mathbb{R}^d.$$

We call an operator D *uniform* if it commutes with all shifts.

We choose a natural number k and a list of k elements of $\mathbb{R}^d$ which we call *neighborhood*:

$$U = \{u_1, \ldots, u_k\} \quad \text{where} \quad u_1, \ldots, u_k \in \mathbb{R}^d.$$

Any neighborhood U whose all elements belong to $\mathbb{Z}^d$ is called a *lattice* neighborhood. Any neighboorhood $U = \{u_1, \ldots, u_k\}$ with k distinct real numbers in ascending order (*i.e.*, $u_1 < u_2 < \cdots < u_k$) is called *ascending*.

Any map $f : M^k \to M$ is called a *transition map*. A neighborhood U and a transition map f with one and the same parameter k determine an operator D by the following rule:

$$(xD)_p = f(x_{p+u_1}, \dots, x_{p+u_k}) \text{ for all } p \in \mathbb{R}^d. \tag{1}$$

We call an operator D *regular* if it is defined by (1) and

$$f(a, \dots, a) = a \text{ for all } a \in M. \tag{2}$$

Notice that a regular operator defined by (1) is uniform and it is monotonic if and only if its transition map f is *monotonic*, that is

$$a_1 \le b_1, \dots, a_k \le b_k \Longrightarrow f(a_1, \dots, a_k) \le f(b_1, \dots, b_k).$$

Henceforth we consider only monotonic regular operators.

1 One-dimensional Background

A configuration $x \in \Omega_1$ is called *increasing* if

$$\forall\, p, q \in \mathbb{R} : p < q \Longrightarrow x_p \le x_q.$$

Analogously, a configuration $x \in \Omega_1$ is called *decreasing* if

$$\forall\, p, q \in \mathbb{R} : p > q \Longrightarrow x_p \le x_q.$$

We say that a configuration is *monotonic* if it is increasing or decreasing. Notice that if x is monotonic, then there are $p_1, p_2 \in \mathbb{R}$ and $\text{limleft}, \text{limright} \in M$ such that

$$x_p = \text{limleft} \text{ for all } p < p_1 \text{ and } x_p = \text{limright} \text{ for all } p > p_2.$$

A monotonic configuration x for which $\text{limleft} \neq \text{limright}$ is called a $(\text{limleft}, \text{limright})$-*ladder*. For any $(\text{limleft}, \text{limright})$-ladder x, we denote

$$\text{Left}(x) = \sup\{\, p \in \mathbb{R} : x_p = \text{limleft}\} \text{ and}$$

$$\text{Right}(x) = \inf\{\, p \in \mathbb{R} : x_p = \text{limright}\}.$$

The real numbers $\text{Left}(x)$ and $\text{Right}(x)$ are called *left barrier* and *right barrier* of x respectively. The non-negative number $\mathit{length}(x) = \text{Right}(x) - \text{Left}(x)$ is called the *length* of the $(\text{limleft}, \text{limright})$-ladder x.

We say that a $(\text{limleft}, \text{limright})$-ladder x is *right-continuous* at $p_0 \in \mathbb{R}$ if there is a positive real number ε such that

$$p_0 < p < p_0 + \varepsilon \Longrightarrow x_p = x_{p_0}.$$

Moreover, a $(\text{limleft}, \text{limright})$-ladder x is said to be *right-continuous* if it is right-continuous at all $p \in \mathbb{R}$. A *left-continuous* ladder is defined analogously. A $(\text{limleft}, \text{limright})$-ladder x with $\mathit{lenght}(x) = 0$, that is either increasing and right-continuous or decreasing and left-continuous is called a $(\text{limleft}, \text{limright})$-*jump* and is denoted by $J_{\text{limleft},\text{limright}}$.

Lemma 1 *Let D be a one-dimensional regular operator and x be a right-continuous increasing* $(\text{limleft}, \text{limright})$*-ladder. Then xD^t is also a right-continuous increasing* $(\text{limleft}, \text{limright})$*-ladder for all natural t.*

From lemma 1 there are real numbers $V_{01}(D), V_{12}(D)$ for which

$$J_{01}D = J_{01}\,\text{Shift}^{V_{01}(D)} \text{ and } J_{12}D = J_{12}\,\text{Shift}^{V_{12}(D)}.$$

The real numbers $V_{01}(D)$ and $V_{12}(D)$ are called the (01)-*rate* and the (12)-*rate* of the one-dimensional operator D respectively.

Lemma 2 *Let D be a one-dimensional regular operator. Then the following limits exist and we denote them thus:*

$$\lim_{t\to\infty} \frac{\text{Left}(J_{02}D^t)}{t} = L_{02}(D)\,, \quad \lim_{t\to\infty} \frac{\text{Right}(J_{02}D^t)}{t} = R_{02}(D)\,,$$

$$\lim_{t\to\infty} \frac{\text{Left}(J_{20}D^t)}{t} = L_{20}(D) \quad \textit{and} \quad \lim_{t\to\infty} \frac{\text{Right}(J_{20}D^t)}{t} = R_{20}(D).$$

Proofs of Lemmas 1 and 2 and other details can be found in [5]. The limits $L_{02}(D)$, $R_{02}(D)$, $L_{20}(D)$ and $R_{20}(D)$ are called *left* (02)*-rate*, *right* (02)*-rate*, *left* (20)*-rate* and *right* (20)*-rate* of the one-dimensional operator D respectively. They are natural generalizations of the Galperin's rates presented in [1, 2].

2 Back to d-dimensional Case

We call by *direction* any point in $\mathbb{R}^d$ whose norm is one. Any direction $\|v\|^{-1}v$ where $v \in \mathbb{Z}^d$ is called a *lattice direction.*

Given a direction δ and neighboorhood $U = \{u_1, \dots, u_k\}$, we call

$$U_\delta = \{u_{\delta 1}, \dots, u_{\delta k}\} = \{\langle u_1, \delta\rangle\,, \langle u_2, \delta\rangle\,, \dots\,, \langle u_k, \delta\rangle\} \tag{3}$$

the δ*-neighborhood.*

Let us fix a d-dimensional regular operator $D : \Omega_d \to \Omega_d$, where $d > 1$. For each direction δ we define a one-dimensional regular operator $D_\delta : \Omega_1 \to \Omega_1$ by the rule

$$(x\, D_\delta)_p = f(x_{p+u_{\delta 1}}, \dots, x_{p+u_{\delta k}}) \quad \text{for all} \quad p \in \mathbb{R}.$$

The *directional* (01)*-rate* and *directional* (12)*-rate* of D along a direction δ are respectively the real numbers $V_{01}(D_\delta)$ and $V_{12}(D_\delta)$ for which

$$J_{01}\, D_\delta = J_{01}\ \text{Shift}^{V_{01}(D_\delta)} \quad \text{and} \quad J_{12}\, D_\delta = J_{12}\ \text{Shift}^{V_{12}(D_\delta)}\,.$$

The *directional left* (02)*-rate*, *directional right* (02)*-rate*, *directional left* (20)*-rate*, and *directional right* (20)*-rate* of D along a direction δ are the following limits:

$$\lim_{t\to\infty} \frac{\text{Left}(J_{02}\, {D_\delta}^t)}{t} = L_{02}(D_\delta), \quad \lim_{t\to\infty} \frac{\text{Right}(J_{02}\, {D_\delta}^t)}{t} = R_{02}(D_\delta),$$

$$\lim_{t\to\infty} \frac{\text{Left}(J_{20}\, {D_\delta}^t)}{t} = L_{20}(D_\delta) \quad \text{and} \quad \lim_{t\to\infty} \frac{\text{Right}(J_{20}\, {D_\delta}^t)}{t} = R_{20}(D_\delta)$$

respectively.

We say that an operator D *erodes* an island x if there is a natural t such that xD^t (the result of t iterative applications of D to x) is "all zeros". We call an operator D an *eroder* if it erodes all islands. The problem of discerning eroders has been studied for years. It was shown [4] that even for very restricted classes of operators the problem of discerning eroders among them is algorithmically unsolvable. Although there are some positive results, most of them pertain to one of these two cases: either the space has dimension one [1, 2] or the set of states has only two elements [10, 11, 12]. The first case beyond these results, which comes to mind is when the dimension of the space is two and the set of states has three elements. For this case a few concrete results have been obtained in [3] and the first article containing general results about this case is [8]. In [1], Galperin have obtained an eroder criteria for discrete one-dimensional cellular automata by using his left and right rates. In [2], Galperin have found a way of computing his rates, but he did not present any detailed routines for computing them. Furthermore, in [5] were proved some erodicity results by using directional Galperin's rates. A natural question is: *How can one actually compute a directional Galperin rate?*

DOI: 10.5540/03.2014.002.01.0084

3 Results

Lemma 3 *Let D be an one-dimensional operator defined by an f and an ascending neighborhood U. Then Algorithm 1 computes V_{01}.*

Algorithm 1 Computing V_{01} routine

Require: k, f, U ▷ function $f : M^k \to M$ and $U = \{u_1, \dots, u_k\}$
$x \leftarrow (0, 0, \dots, 0)$ ▷ $x \in \{0,1\}^k$
for $i \leftarrow 1, k$ **do**
    $x(k - i + 1) \leftarrow 1$
    if $f(x) = 1$ **then**
        $i_{01} = k - i + 1$
    end if
end for
$V_{01} = -u_{i_{01}}$

Computations of V_{12}, V_{10} and V_{21} are analogous.

Lemma 4 *If $V_{01} < V_{12}$, then $L_{02} = V_{01}$ and $R_{02} = V_{12}$.*

Lemma 5 *Let D be an one-dimensional operator defined by an f and an ascending lattice neighborhood U. Then Algorithm 2 computes L_{02} and R_{02}.*

Algorithm 2 Computing R_{02} and L_{02} routine

Require: k, f, U ▷ function $f : M_2^k \to M_2$ and $U = \{u_1, u_2, \dots, u_k\}$
Computing V_{01} routine
Computing V_{12} routine
if $V_{01} \leq V_{12}$ **then**
    $L_{02} = V_{01}$ and $R_{02} = V_{12}$
else
    $C = 0$ and Ladder$(C) = J_{02}$
    while Lenght(Ladder(C)) do not repeat **do**
        Ladder$(C+1)$= (D Ladder(C))
        $C = C + 1$
    end while
    T = steps to repeat, $L_{02} = R_{02} = (\,\text{Right}(\text{Ladder}(C)) - \text{Right}(\text{Ladder}(C - T))\,)/T$
end if

Now, suppose that $U = \{u_1, \dots, u_k\}$ with k distinct real numbers is not an ascending neighborhood. Let us denote by $\tilde{U} = \{\tilde{u}_1, \dots, \tilde{u}_k\}$ the ascending neighborhood obtained by sorting the elements of U in the ascending order. Let us denote by $\rho : \{1, 2, \dots, k\} \to \{1, 2, \dots, k\}$ the permutation for which

$$u_i = \tilde{u}_{\rho(i)} \quad \text{for all} \quad i \in \{1, 2, \dots, k\} \ .$$

Consider the map $\phi : M^k \to M^k$ given by

$$\phi(a_1, \dots, a_k) = (a_{\rho(1)}, \dots, a_{\rho(k)})\,. \tag{4}$$

Lemma 6 *Let D and $\tilde{D}$ be one-dimensional operators defined by f, U and $\tilde{f} = f \circ \phi$, $\tilde{U}$ respectively. Then $D = \tilde{D}$.*

DOI: 10.5540/03.2014.002.01.0084

At last, suppose that $U = \{u_1, \ldots, u_k\}$ has not k distinct elements. Algorithm 3 give us a neighborhood $U^* = \{u_1^*, \ldots, u_{k^*}^*\}$ with k^* distinct elements and a map π. Let us define a transition map $f^* : M^{k^*} \to M$ by

$$f^*(a_1, \ldots, a_{k^*}) = f(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(k)})\,.$$

Algorithm 3 Obtaining π and U^* routine

Require: k, U $\triangleright$ $U = \{u_1, u_2, \ldots, u_k\}$

$u_1^* = u_1$.
$\pi(1) = 1$.
for $j \leftarrow 2, k$ **do**
    $\tau \leftarrow u_j$
    $l \leftarrow 1$
    while $\tau \neq u_l^*$ **do**
        $l \leftarrow l + 1$
    end while
    $u_l^* \leftarrow \tau$
    $\pi(j) \leftarrow l$
end for
return π, k^* and U^*

Lemma 7 *Let D and D^* be one-dimensional operators defined by f, U and f^*, U^* respectively. Then $D = D^*$.*

Theorem 1 *Let D be one-dimensional operator defined by f and lattice neighborhood U. Then Algorithm 4 computes L_{02} and R_{02}.*

Algorithm 4 Computing Galperin's rates routine

Require: k, f, U $\triangleright$ function $f : M^k \to M$ and $U = \{u_1, u_2, \ldots, u_k\}$

1: Obtaining π and U^* routine with k and U
2: $f^*(a_1, \ldots, a_{k^*}) = f(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(k)})$
3: Sorting the elements of U^* in the ascending order returning $\tilde{U}^*$ and ρ
4: $\phi(a_1, \ldots, a_{k^*}) = (a_{\rho(1)}, \ldots, a_{\rho(k^*)})$
5: $\tilde{f}^* = f^* \circ \phi$
6: Computing V_{01} and V_{12} routine with $\tilde{f}^*$ and $\tilde{U}^*$
7: Computing R_{02} and L_{02} routine with $\tilde{f}^*$ and $\tilde{U}^*$

Corollary 1 *Let D be a d-dimendional operator defined by a boolean monotone function f and a neighborhood U. Let δ be a direction. The first 5 steps of Algorithm 4 with k, f_δ and U_δ returns $\tilde{f}_\delta^*$ and $\tilde{U}_\delta^*$ and Algorithm 1 with $\tilde{f}_\delta^*$ and $\tilde{U}_\delta^*$ returns the directional (01)-rate of D along direction δ.*

Corollary 2 *Let D be a d-dimensional operator defined by f and lattice neighborhood U. Given $v \in \mathbb{Z}^d$, consider $W = \{\langle u_1, v\rangle, \ldots, \langle u_d, v\rangle\}$. Let D' be the one-dimensional operator defined by f and W. Algorithm 4 with f and W returns $V_{01}(D')$, $V_{12}(D')$, $L_{02}(D')$, $R_{02}(D')$, and $\|v\|^{-1}V_{01}(D')$, $\|v\|^{-1}V_{12}(D')$, $\|v\|^{-1}L_{02}(D')$, $\|v\|^{-1}R_{02}(D')$ are the directional Galperin's rates of D along the lattice direction $\|v\|^{-1}v$.*

DOI: 10.5540/03.2014.002.01.0084

4 Conclusions

In order to test our algorithms we have developed and implemented a routine that give us random examples of monotone transition maps [6]. Algorithm 4 will facilitate the inductive reasoning concerning the conjecture presented in [9]. There is an equivalent formulation for Toom's erodicity criteria [10] by using a finite set of directional (01)-rates [7]. Once erodicity of two-state cellular automata could be answered by using just a finite set of directional (01)-rates, a priori, it is plausible to believe that we can obtain similar results to those presented in [8] by using just a finite set of directional right (02)-rates too. Corollary 2 will be useful in investigating that belief.

Acknowledgments

L. H. de Santana and A. Toom thank CNPq-Brazil for the financial support.

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