On monotone multivalued functions: generating and estimating Dedekind numbers

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Abstract: Monotone multivalued functions have been widely employed in discrete mathematics, multivalued logics and reliability theory, in special its particular case of monotone boolean functions. However, our interest in them came from the kind of monotone cellular automata studied in [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13], where monotone multivalued functions play a fundamental role. At first, we present an algorithm that generates random examples of monotone multivalued functions. Our second algorithm provide an estimative for the number of monotone multivalued functions, which shall be called here Dedekind number. Our first algorithm provided examples that were used for testing routines that compute Galperin’s rates [9]. Moreover, it will facilitate the inductive reasoning concerning monotone cellular automata.

Keywords: monotone boolean function, monotone multivalued function, Dedekind number

1 Introduction

Let \( m \) and \( k \) be natural numbers. The set \( M_m = \{0, 1, \ldots, m\} \) is called the set of states. Let \( v=(v_1,\ldots,v_k) \) and \( u=(u_1,\ldots,u_k) \) be elements of \( M_m^k \). We denote

\[ v < u \iff v_1 \leq u_1, \ldots, v_k \leq u_k. \]

Notice that \(<\) defines a partial order.

Any function \( f : M_m^k \to M_m \) is called a multivalued. A multivalued function \( f : M_m^k \to M_m \) is called monotone if

\[ \forall v, u \in M_m^k : v < u \Rightarrow f(v) \leq f(u). \] (1)

Notice that a multivalued function \( f : M_m^k \to M_m \) is monotone if and only if it satisfies the following set of integer inequalities:

\[ 0 \leq f(a_1, a_2, \ldots, a_k) \leq m \quad \text{for all} \quad a_1, a_2, \ldots, a_k \in M_m \]
\[ 0 \leq f(a_1 + 1, a_2, \ldots, a_k) - f(a_1, a_2, \ldots, a_k) \quad \text{for all} \quad a_2, a_3, \ldots, a_k \in M_m, a_1 \in M_{m-1} \]
\[ 0 \leq f(a_1, a_2 + 1, \ldots, a_k) - f(a_1, a_2, \ldots, a_k) \quad \text{for all} \quad a_1, a_3, \ldots, a_k \in M_m, a_2 \in M_{m-1} \] (2)

\[ \vdots \]
\[ 0 \leq f(a_1, a_2, \ldots, a_k + 1) - f(a_1, a_2, \ldots, a_k) \quad \text{for all} \quad a_2, \ldots, a_{k-1} \in M_m, a_k \in M_{m-1} \]

Sometimes it is also assumed that

\[ f(a, \ldots, a) = a \quad \text{for all} \quad a \in M_m. \] (3)

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[1, 2, 4, 7, 8, 9, 10, 11, 12, 13], where monotone multivalued functions satisfying assumption (3) play a fundamental role.

Counting the number of monotone functions from $M^k_1$ to $M_1$ is a classical problem. This number is known as Dedekind number. Here the number of monotone multivalued functions from $M^k_m$ to $M_m$ shall be also called Dedekind number.

Our questions are:

1. Given $k$ and $m$, how can we can generate a random example of monotone multivalued function?
2. Given multivalued function, how can we estimate Dedekind numbers for multivalued monotone functions?

## 2 Generating

The map $d : M^k_m \times M^k_m \rightarrow \mathbb{R}_+$ given by

$$d(v, v') = \sum_{i=1}^{k} |v_i - v'_i|$$

is a metric. The simple oriented graph $H_{m,k}$ where the set of vertices is $M^k_m$ and the set of edges is

$$E_{m,k} = \{ (v, v') \in M^k_m \times M^k_m : d(v, v') = 1, d(v, 0) < d(v', 0) \}$$

is called Hasse diagram of $M^k_m$. The adjacency matrix of graph $H_{m,k}$ is denoted by $A_{m,k}$.

A partial order $\prec^*$ on the set $M^k_m$ in an extension of another partial order $\prec$ on the set $M^k_m$ if

$$\forall v, u \in M^k_m : v \prec u \Rightarrow v \prec^* u.$$ 

A linear extension is an extension that is also a total order.

### Algorithm 1 Random generator of monotone multivalued functions

**Require:** $k, m$
1. Obtaining $A_{m,k}$ routine.
2. Raffling a linear extension routine with $A_{m,k}$.
3. Third routine with the outcome from the previous routine.

The Raffling a linear extension routine was obtained by modifying the topological ordering algorithm [5] and using some ideas presented in [6]. This routine requires the partial order $\prec$ on the set $M^k_m$ represented by the adjacency matrix $A_{m,k}$ and returns a random example of linear extension $\prec^*$ of $\prec$ on the set $M^k_m$.

Let us present the idea of the Third routine. After the second routine we have

$$(0, 0, \ldots, 0) = p_1 \prec^* p_2 \prec^* \cdots \prec^* p_{(m+1)k} = (m, m, \ldots, m),$$

where $p_j$ denotes the $j$-th element according the total order and $j \in \{1, 2, \ldots, (m + 1)^k\}$. Suppose that the outcome of an uniform discrete distribution in the integer interval $[1, (m + 1)^k]$ is $j_1$. Then $f(p_{j_1}) = 0$ for all $j < j_1$ and $f(p_{j_1}) = 1$. If $j_1 = (m + 1)^k$, then it is over. Otherwise, suppose that the outcome of an uniform discrete distribution in the interval $[j_1 + 1, (m + 1)^k]$ is $j_2$. Then $f(p_{j_2}) = 1$ for all $j_1 < j < j_2$ and $f(p_{j_2}) = 2$. If $j_2 = (m + 1)^k$, then it is over. Otherwise, suppose that the outcome of an uniform discrete distribution in the integer interval $[j_2 + 1, (m + 1)^k]$ is $j_3$ and so on.

**Theorem 1** Given $k$ and $m$, the outcome of Algorithm 1 is a random variable whose sample space is the set of all monotone multivalued functions from $M^k_m$ to $M_m$. 

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We have implemented Algorithm 1 and a modified version of Algorithm 1 whose outcome is a random variable where its sample space is the set of all monotone multivalued functions from $M_m^k$ to $M_m$ satisfying (3).

3 Estimating Dedekind number

Consider the set $F_{m,k}$ of all multivalued functions from $M_m^k$ to $M_m$. We know that $F_{m,k}$ has $(m+1)^{(m+1)^k}$ elements. However, how many among these elements are monotone?

Let $n$ be a natural number. Algorithm 2 estimates the proportion of monotone multivalued functions in the set $F_{m,k}$. The system of integer inequalities (2) can be used for testing monotonicity of multivalued function $f_i$ in Algorithm 2.

**Algorithm 2 Estimating proportion routine**

**Require:** $k, m, n$

1: for $j \leftarrow 1, n$ do
2:   $s \leftarrow 0$
3:   Generate a random example of multivalued function $f_j$
4:   if $f_j$ is monotone then
5:     $s \leftarrow s + 1$
6:   end if
7: end for
8: return $p_n = \frac{n-1}{n} s$

Lemma 1 The sequence \( \{p_n (m+1)^{(m+1)^k}\}_{n \in \mathbb{N}} \) converges in probability to the Dedekind number.

4 Conclusions

Algorithm 2 is not implemented yet. The modified version of Algorithm 1 provided examples that were used for testing routines that compute Galperin’s rates [9]. Moreover, Algorithm 1 will facilitate the inductive reasoning concerning monotone cellular automata.

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Referências


