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On monotone multivalued functions: generating and estimating Dedekind numbers

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Abstract: Monotone multivalued functions have been widely employed in discrete mathematics, multivalued logics and reliability theory, in special its particular case of monotone boolean functions. However, our interest in them came from the kind of monotone cellular automata studied in [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13], where monotone multivalued functions play a fundamental role. At first, we present an algorithm that generates random examples of monotone multivalued functions. Our second algorithm provide an estimative for the number of monotone multivalued functions, which shall be called here Dedekind number. Our first algorithm provided examples that were used for testing routines that compute Galperin's rates [9]. Moreover, it will facilitate the inductive reasoning concerning monotone cellular automata.

Keywords: *monotone boolean function, monotone multivalued function, Dedekind number*

1 Introduction

Let m and k be natural numbers. The set $M_m = \{0, 1, \dots, m\}$ is called the *set of states*. Let $v = (v_1, \dots, v_k)$ and $u = (u_1, \dots, u_k)$ be elements of M_m^k. We denote

$$v \prec u \Leftrightarrow v_1 \le u_1 \,, \, \dots \,, \, v_k \le u_k \,.$$

Notice that $\prec$ defines a partial order.

Any function $f : M_m^k \to M_m$ is called a *multivalued*. A multivalued function $f : M_m^k \to M_m$ is called *monotone* if

$$\forall v, u \in M_m^k \,:\, v \prec u \Rightarrow f(v) \le f(u) \,. \tag{1}$$

Notice that a multivalued function $f : M_m^k \to M_m$ is monotone if and only if it satisfies the following set of integer inequalities:

$$\begin{array}{rll} 0 \le f(a_1, a_2, \dots, a_k) \le m & \text{for all} & a_1, a_2, \dots, a_k \in M_m \\ 0 \le f(a_1 + 1, a_2, \dots, a_k) - f(a_1, a_2, \dots, a_k) & \text{for all} & a_2, a_3, \dots, a_k \in M_m, a_1 \in M_{m-1} \\ 0 \le f(a_1, a_2 + 1, \dots, a_k) - f(a_1, a_2, \dots, a_k) & \text{for all} & a_1, a_3, \dots, a_k \in M_m, a_2 \in M_{m-1} \\ & \vdots & \\ 0 \le f(a_1, a_2, \dots, a_k + 1) - f(a_1, a_2, \dots, a_k) & \text{for all} & a_2, \dots, a_{k-1} \in M_m, a_k \in M_{m-1} \end{array} \tag{2}$$

Sometimes it is also assumed that

$$f(a, \dots, a) = a \quad \text{for all} \quad a \in M_m \,. \tag{3}$$

Monotone multivalued functions have been widely employed in discrete mathematics, multivalued logics and reliability theory, in special its particular case of monotone boolean functions. However, our interest in them came from the kind of cellular automata studied in

[1, 2, 4, 7, 8, 9, 10, 11, 12, 13], where monotone multivalued functions satisfying assumption (3) play a fundamental role.

Counting the number of monotone functions from M_1^k to M_1 is a classical problem. This number is known as *Dedekind number*. Here the number of monotone multivalued functions from M_m^k to M_m shall be also called Dedekind number.

Our questions are:

1. *Given k and m, how can we can generate a random example of monotone multivalued function?*
2. *Given multivalued function, how can we estimate Dedekind numbers for multivalued monotone functions?*

2 Generating

The map $d : M_m^k \times M_m^k \to \mathbb{R}_+$ given by

$$d(v, v') = \sum_{i=1}^{k} |v_i - v'_i|$$

is a metric. The simple oriented graph $H_{m,k}$ where the set of vertices is M_m^k and the set of edges is

$$E_{m,k} = \Big\{ (v, v') \in M_m^k \times M_m^k \,:\, d(v, v') = 1,\, d(v, 0) < d(v', 0) \Big\}$$

is called *Hasse diagram* of M_m^k. The adjancency matrix of graph $H_{m,k}$ is denoted by $\mathcal{A}_{m,k}$.

A partial order $\prec^*$ on the set M_m^k in an *extension* of another partial order $\prec$ on the set M_m^k if

$$\forall v, u \in M_m^k \,:\, v \prec u \Rightarrow v \prec^* u\,.$$

A *linear extension* is an extension that is also a total order.

Algorithm 1 Random generator of monotone multivalued functions

Require: k, m

1: Obtaining $\mathcal{A}_{m,k}$ routine.
2: Raffling a linear extension routine with $\mathcal{A}_{m,k}$.
3: Third routine with the outcome from the previous routine.

The *Raffling a linear extension routine* was obtained by modifying the *topological ordering algorithm* [5] and using some ideas presented in [6]. This routine requires the partial order $\prec$ on the set M_m^k represented by the adjacency matrix $\mathcal{A}_{m,k}$ and returns a random example of linear extension $\prec^*$ of $\prec$ on the set M_m^k.

Let us present the idea of the *Third routine*. After the second routine we have

$$(0, 0, \ldots, 0) = p_1 \prec^* p_2 \prec^* \cdots \prec^* p_{(m+1)^k} = (m, m, \ldots, m)\,,$$

where p_j denotes the j-th element according the total order and $j \in \big\{1, 2, \ldots, (m+1)^k\big\}$. Suppose that the outcome of an uniform discrete distribution in the integer interval $[1, (m+1)^k]$ is j_1. Then $f(p_j) = 0$ for all $j < j_1$ and $f(p_{j_1}) = 1$. If $j_1 = (m+1)^k$, then it is over. Otherwise, suppose that the outcome of an uniform discrete distribution in the interval $[j_1 + 1, (m+1)^k]$ is j_2. Then $f(p_j) = 1$ for all $j_1 < j < j_2$ and $f(p_{j_2}) = 2$. If $j_2 = (m+1)^k$, then it is over. Otherwise, suppose that the outcome of an uniform discrete distribution in the integer interval $[j_2 + 1, (m+1)^k]$ is j_3 and so on.

Theorem 1 *Given k and m, the outcome of Algorithm 1 is a random variable whose sample space is the set of all monotone multivalued functions from M_m^k to M_m.*

We have implemented Algorithm 1 and a modified version of Algorithm 1 whose outcome is a random variable where its sample space is the set of all monotone multivalued functions from M_m^k to M_m satisfying (3).

3 Estimating Dedekind number

Consider the set $\mathcal{F}_{m,k}$ of all multivalued functions from M_m^k to M_m. We know that $\mathcal{F}_{m,k}$ has $(m+1)^{(m+1)^k}$ elements. However, how may among these elements are monotone?

Let n be a natural number. Algorithm 2 estimates the proportion of monotone multivalued functions in the set $\mathcal{F}_{m,k}$. The system of integer inequalities (2) can be used for testing

Algorithm 2 Estimating proportion routine

Require: k, m, n
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for j ← 1, n do
    s ← 0
    Generate a random example of multivalued function f_j
    if f_j is monotone then
        s ← s + 1
    end if
end for
return p_n = n^{-1} s
```

monotonicity of multivalued function f_i in Algorithm 2.

Lemma 1 *The sequence* $\{p_n\,(m+1)^{(m+1)^k}\}_{n\in\mathbb{N}}$ *converges in probability to the Dedekind number.*

4 Conclusions

Algorithm 2 is not implemented yet. The modified version of Algorithm 1 provided examples that were used for testing routines that compute Galperin's rates [9]. Moreover, Algorithm 1 will facilitate the inductive reasoning concerning monotone cellular automata.

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