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# A sufficient condition for reciprocal Jacobi triangles

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**Abstract**. Given a triangle $ABC$ let $A'B'C'$ be a Jacobi triangle for $ABC$. When $\angle BA'C' = \angle CA'B' = \alpha'$, $\angle AB'C' = \angle CB'A' = \beta'$ and $\angle AC'B' = \angle BC'A' = \gamma'$, the triangle $ABC$ is a Jacobi triangle for $A'B'C'$. In this case we say that $ABC$ and $A'B'C'$ are reciprocal Jacobi triangles. In 2015, G.T. Vickers presented a necessary condition for two triangles to be reciprocal, but the question whether that condition was also sufficient remained open. In this work we prove it by using basically trigonometric relations.

**Keywords**. Jacobi's Theorem, Triangle, Geometry.

## 1 Introduction

On the sides of a given (arbitrary) triangle $ABC$ construct three similar isosceles triangles $ABC'$, $BCA'$, $ACB'$, exterior to $ABC$, such that the angular measure $\angle BAC' = \angle ABC' = \angle BCA' = \angle CBA' = \angle ACB' = \angle CAB'$. Ludwig Kiepert showed that in this case the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent at a point $K$, which is called the Kiepert point of the triangle $ABC$. Carl Friedrich Andreas Jacobi (1795-1855), a German mathematician who devoted himself to study the triangular geometry, generalized the Kiepert's construction as it follows:

**Theorem 1.1.** *(Jacobi's theorem) Let $ABC$ be an arbitrary triangle, $A'$, $B'$, $C'$ points such that $\angle C'AB = \angle CAB' = \alpha$, $\angle A'BC = \angle C'BA = \beta$ and $\angle B'CA = \angle A'CB = \gamma$. The lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent at a point $K$.*

The triangle $A'B'C'$ and the point $K$ are called the Jacobi triangle for the triangle $ABC$ with respect to the angles $\alpha, \beta, \gamma$ and the Jacobi point for the triangle $A'B'C'$, respectively. We illustrate one Jacobi triangle in Figure 1. We consider $\alpha, \beta, \gamma \in (0, \pi)$, but they can be naturally extended to the interval $(0, 2\pi)$.

There are many proofs for Theorem 1.1. One idea to prove the existence of $K$ is to apply the following important result of the geometry of triangles:

(Ceva's theorem) *The three lines containing the vertices $A, B, C$ of the triangle $ABC$ and intersecting the opposite sides in points $P$, $Q$, and $R$, respectively, are concurrent if and only if* $\dfrac{BR}{RA} \cdot \dfrac{CP}{BP} \cdot \dfrac{AQ}{QC} = 1$.

This strategy was used in [2] to prove Theorem 1.1. Here we present a scheme for it. The details can be viewed in [1].

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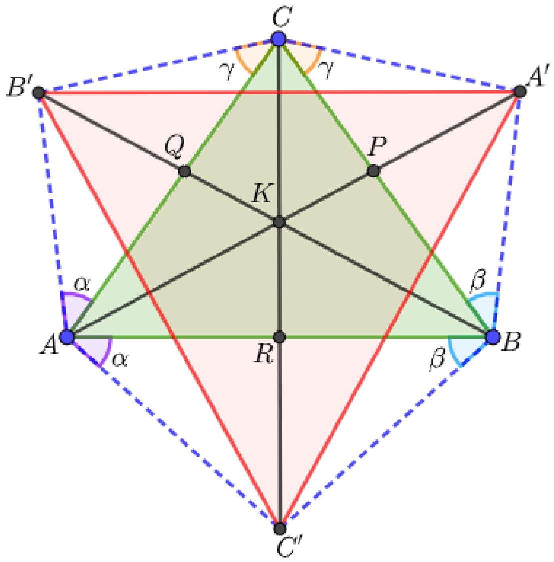


Figure 1: The Jacobi triangle $A'B'C'$ for the triangle $ABC$.

*Proof.* As illustrated in Figure 1, let $\{P\} = \overleftrightarrow{AA'} \cap \overleftrightarrow{BC}$, $\{Q\} = \overleftrightarrow{BB'} \cap \overleftrightarrow{AC}$ and $\{R\} = \overleftrightarrow{CC'} \cap \overleftrightarrow{AB}$. In Figure 2 we see a schematic proof that $\dfrac{BR}{RA} \cdot \dfrac{CP}{BP} \cdot \dfrac{AQ}{QC} = 1$.

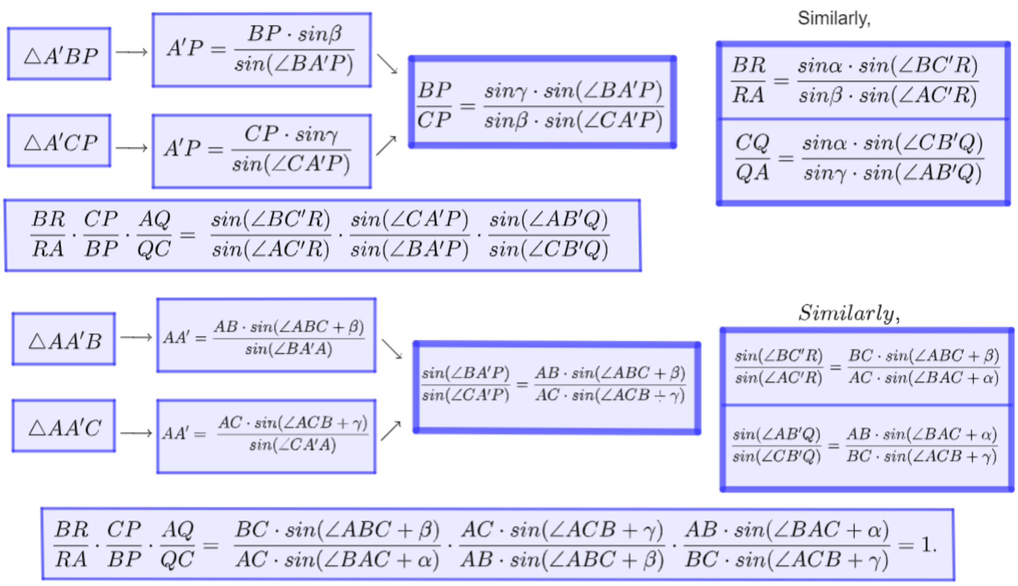


Figure 2: Scheme for Jacobi's theorem proof.

From Ceva's theorem it follows that $\overleftrightarrow{BB'} \cap \overleftrightarrow{AA'} \cap \overleftrightarrow{CC'} = \{K\}$. □

 

## 2 Reciprocal Jacobi triangles

**Definition 2.1.** *Let $ABC$ be any triangle and $A'B'C'$ be the Jacobi triangle for $ABC$ with respect to the angles $\alpha, \beta, \gamma$ constructed on the vertices $A, B, C$, respectively. If the Jacobi triangle with angles $\alpha', \beta'$ and $\gamma'$ for the triangle $A'B'C'$ coincides with $ABC$, we say that $ABC$ and $A'B'C'$ are reciprocal Jacobi triangles.*

In Figure 3 we can see two reciprocal Jacobi triangles.

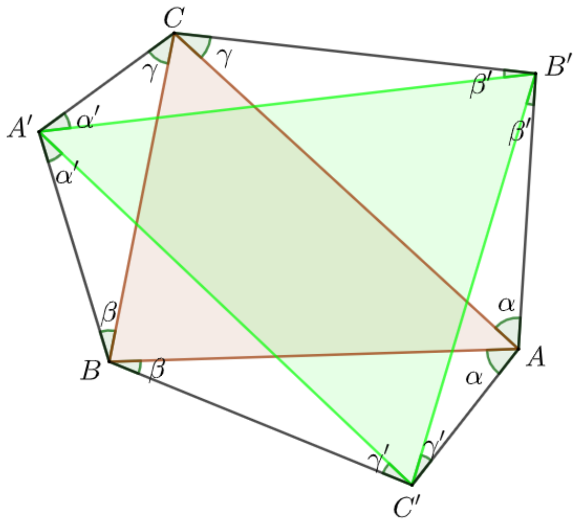


Figure 3: $ABC$ and $A'B'C'$ are reciprocal Jacobi triangles.

In the conditions of Definition 2.1, Vickers has shown in [2] that if $ABC$ and $A'B'C'$ are reciprocal Jacobi triangles then

$$\frac{sin(\angle BAC + 2\alpha)}{sin(\angle BAC)} = \frac{sin(\angle ABC + 2\beta)}{sin(\angle ABC)} = \frac{sin(\angle ACB + 2\gamma)}{sin(\angle ACB)}.$$

The converse of that result was an open question since 2015. We proved it during Sandra Lieven's master's studies and present it bellow.

**Proposition 2.1.** *Let $ABC$ be any triangle and $A'B'C'$ its Jacobi triangle with angles $\alpha, \beta, \gamma$ constructed on $A, B, C$, respectively, such that*

$$\frac{sin(\angle BAC + 2\alpha)}{sin(\angle BAC)} = \frac{sin(\angle ABC + 2\beta)}{sin(\angle ABC)} = \frac{sin(\angle ACB + 2\gamma)}{sin(\angle ACB)} = \mu,$$

*where $\mu$ is a real constant. Then $ABC$ and $A'B'C'$ are reciprocal Jacobi triangles.*

*Proof.* Let $\angle CA'B' = \alpha', \angle BA'C' = \theta, \angle AB'C' = \beta', \angle CB'A' = \phi, \angle BC'A' = \gamma'$ and $\angle AC'B' = \psi$, as illustrated in Figure 4.

By the law of sines for triangles $ABC'$ and $AB'C'$ we have

$$\frac{AC'}{\sin\beta} = \frac{BC'}{\sin\alpha} = \frac{AB}{\sin(\pi - \alpha - \beta)} = \frac{AB}{\sin(\alpha + \beta)} \quad \text{and} \quad \frac{AC'}{\sin\beta'} = \frac{AB'}{\sin\psi}.$$

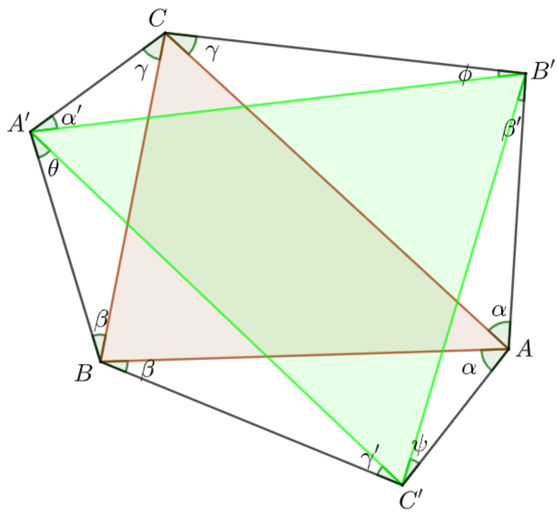


Figure 4: We need to prove that $\theta = \alpha', \phi = \beta'$ and $\psi = \gamma'$.

Consequently,

$$\frac{AB \cdot \sin\beta}{\sin(\alpha+\beta)} = AC' = \frac{AB' \cdot \sin\beta'}{\sin\psi},$$

so that

$$\frac{AB \cdot \sin\beta}{\sin(\alpha+\beta)} \cdot \frac{1}{\sin\beta'} = \frac{AB'}{\sin\psi}. \tag{1}$$

From the law of sines for triangle $AB'C$ it follows that

$$AB' = \frac{AC \cdot \sin\gamma}{\sin(\alpha+\gamma)}. \tag{2}$$

From (1) and (2) we get

$$\frac{AB \cdot \sin\beta}{\sin(\alpha+\beta)} \cdot \frac{1}{\sin\beta'} = \frac{AC \cdot \sin\gamma}{\sin(\alpha+\gamma)} \cdot \frac{1}{\sin\psi}.$$

From triangle $AB'C'$ we have $\psi = \pi - \angle BAC - 2\alpha - \beta'$. Then

$$\frac{AB \cdot \sin\beta}{\sin(\alpha+\beta)} \cdot \frac{1}{\sin\beta'} = \frac{AC \cdot \sin\gamma}{\sin(\alpha+\gamma) \cdot \sin(\angle BAC + 2\alpha + \beta')} \Longrightarrow$$

$$\frac{\sin(\angle BAC + 2\alpha + \beta')}{\sin\beta'} = \frac{AC \cdot \sin\gamma \cdot \sin(\alpha+\beta)}{AB \cdot \sin\beta \cdot \sin(\alpha+\gamma)} \Longrightarrow$$

$$\frac{\sin(\angle BAC + 2\alpha) \cdot \cos\beta' + \sin\beta' \cdot \cos(\angle BAC + 2\alpha)}{\sin\beta'} = \frac{AC \cdot \sin\gamma \cdot \sin(\alpha+\beta)}{AB \cdot \sin\beta \cdot \sin(\alpha+\gamma)} \Longrightarrow$$

$$\sin(\angle BAC + 2\alpha) \cdot \cot\beta' + \cos(\angle BAC + 2\alpha) = \frac{AC \cdot \sin\gamma \cdot \sin(\alpha+\beta)}{AB \cdot \sin\beta \cdot \sin(\alpha+\gamma)} \Longrightarrow$$

$$\cot\beta' = \frac{AC \cdot \sin\gamma \cdot \sin(\alpha+\beta)}{AB \cdot \sin\beta \cdot \sin(\alpha+\gamma) \cdot \sin(\angle BAC + 2\alpha)} - \cot(\angle BAC + 2\alpha).$$

 

Since

$$\frac{\sin(\angle BAC + 2\alpha)}{\sin(\angle BAC)} = \mu,$$

we obtain

$$\mu \cdot \cot \beta' = \frac{AC \cdot \sin\gamma \cdot \sin(\alpha+\beta)}{AB \cdot \sin\beta \cdot \sin(\alpha+\gamma) \cdot \sin(\angle BAC)} - \frac{\cos(\angle BAC + 2\alpha)}{\sin(\angle BAC)}. \tag{3}$$

From triangle $ABC$,

$$\frac{AC}{AB} = \frac{\sin(\angle ABC)}{\sin(\angle ACB)}$$

and $\angle ABC = \pi - \angle ACB - \angle BAC$. Thus $\sin(\angle ABC) = \sin(\angle BAC + \angle ACB)$ and

$$\frac{AC}{AB} = \frac{\sin(\angle BAC + \angle ACB)}{\sin(\angle ACB)} = \frac{\sin(\angle BAC)\cdot\cos(\angle ACB) + \sin(\angle ACB)\cdot\cos(\angle BAC)}{\sin(\angle ACB)} \Longrightarrow$$

$$\frac{AC}{AB} = \sin(\angle BAC)\cdot\cot(\angle ACB) + \cos(\angle BAC). \tag{4}$$

From (3) and (4) we have that $\mu \cdot \cot\beta' =$

$$(\sin(\angle BAC)\cdot\cot(\angle ACB) + \cos(\angle BAC)) \cdot \frac{\sin\gamma\cdot\sin(\alpha+\beta)}{\sin\beta\cdot\sin(\alpha+\gamma)\cdot\sin(\angle BAC)} - \frac{\cos(\angle BAC+2\alpha)}{\sin(\angle BAC)}.$$

So

$$\mu\cdot\cot\beta' = (\cot(\angle ACB) + \cot(\angle BAC))\cdot\frac{\sin\gamma\cdot\sin(\alpha+\beta)}{\sin\beta\cdot\sin(\alpha+\gamma)} - \frac{\cos(\angle BAC+2\alpha)}{\sin(\angle BAC)}. \tag{5}$$

But

$$-\frac{\cos(\angle BAC+2\alpha)}{\sin(\angle BAC)} = \frac{-\cos(\angle BAC)\cdot\cos 2\alpha + \sin(\angle BAC)\cdot\sin 2\alpha}{\sin(\angle BAC)} \Longrightarrow$$

$$-\frac{\cos(\angle BAC+2\alpha)}{\sin(\angle BAC)} = -\cot(\angle BAC)\cdot\cos 2\alpha + \sin 2\alpha \Longrightarrow$$

$$-\frac{\cos(\angle BAC+2\alpha)}{\sin(\angle BAC)} = \frac{1}{\sin 2\alpha}\cdot\left(\sin^2 2\alpha - \cot(\angle BAC)\cdot\cos 2\alpha\cdot\sin 2\alpha\right) \Longrightarrow$$

$$-\frac{\cos(\angle BAC+2\alpha)}{\sin(\angle BAC)} = \frac{1}{\sin 2\alpha}\cdot\left(1 - \cos^2 2\alpha - \frac{\cos(\angle BAC)}{\sin(\angle BAC)}\cdot\cos 2\alpha\cdot\sin 2\alpha\right) \Longrightarrow$$

$$-\frac{\cos(\angle BAC+2\alpha)}{\sin(\angle BAC)} = \frac{1}{\sin 2\alpha}\cdot\left(1 - \cos 2\alpha\left(\cos 2\alpha + \sin 2\alpha\cdot\frac{\cos(\angle BAC)}{\sin(\angle BAC)}\right)\right) \Longrightarrow$$

$$-\frac{\cos(\angle BAC+2\alpha)}{\sin(\angle BAC)} = \frac{1}{\sin 2\alpha}\cdot\left(1 - \cos 2\alpha\left(\frac{\sin(\angle BAC)\cdot\cos 2\alpha + \sin 2\alpha\cdot\cos(\angle BAC)}{\sin(\angle BAC)}\right)\right).$$

Thus

$$-\frac{\cos(\angle BAC+2\alpha)}{\sin(\angle BAC)} = \frac{1}{\sin 2\alpha}\cdot(1 - \cos 2\alpha\cdot\mu) = -\frac{(\mu\cdot\cos 2\alpha - 1)}{\sin 2\alpha}. \tag{6}$$

 

By replacing (6) in (5) we get

$$\mu \cdot \cot \beta' = (\cot(\angle ACB) + \cot(\angle BAC)) \cdot \frac{\sin\gamma \cdot \sin(\alpha+\beta)}{\sin\beta \cdot \sin(\alpha+\gamma)} - \frac{(\mu \cdot \cos 2\alpha - 1)}{\sin 2\alpha}. \tag{7}$$

Analogously

$$\mu \cdot \cot \phi = (\cot(\angle ACB) + \cot(\angle BAC)) \cdot \frac{\sin\alpha \cdot \sin(\gamma+\beta)}{\sin\beta \cdot \sin(\alpha+\gamma)} - \frac{(\mu \cdot \cos 2\gamma - 1)}{\sin 2\gamma}. \tag{8}$$

If $\alpha = \gamma$ then by (7) and (8) we have $\mu \cdot \cot \beta' = \mu \cdot \cot \phi$. Therefore $\beta' = \phi$ for $\beta', \phi \in (0, \pi)$. We now suppose that $\alpha \neq \gamma$. Since

$$\mu = \frac{\sin(\angle BAC + 2\alpha)}{\sin(\angle BAC)} = \frac{\sin(\angle BAC) \cdot \cos 2\alpha + \sin 2\alpha \cdot \cos(\angle BAC)}{\sin(\angle BAC)} = \cos 2\alpha + \sin 2\alpha \cdot \cot(\angle BAC),$$

thus

$$\cot(\angle BAC) = \frac{\mu}{\sin 2\alpha} - \cot 2\alpha. \tag{9}$$

In a similar way,

$$\cot(\angle ACB) = \frac{\mu}{\sin 2\gamma} - \cot 2\gamma. \tag{10}$$

From (9) and (10) we get

$$\cot(\angle BAC) + \cot(\angle ACB) = \frac{\mu}{\sin 2\alpha} - \cot 2\alpha + \frac{\mu}{\sin 2\gamma} - \cot 2\gamma =$$

$$= \frac{\mu \cdot (\sin 2\alpha + \sin 2\gamma) - \cos 2\alpha \cdot \sin 2\gamma - \cos 2\gamma \cdot \sin 2\alpha}{\sin 2\alpha \cdot \sin 2\gamma} = \frac{\mu \cdot (\sin 2\alpha + \sin 2\gamma) - \sin(2\alpha + 2\gamma)}{\sin 2\alpha \cdot \sin 2\gamma}.$$

Thus

$$\sin 2\alpha \cdot \sin 2\gamma \cdot (\cot(\angle BAC) + \cot(\angle ACB)) = \mu \cdot (\sin 2\alpha + \sin 2\gamma) - \sin(2\alpha + 2\gamma). \tag{11}$$

But $\cot \beta' = \cot \phi \iff \mu \cdot \cot \beta' = \mu \cdot \cot \phi \iff$

$$(\cot(\angle ACB) + \cot(\angle BAC)) \cdot \frac{\sin\gamma \cdot \sin(\alpha+\beta)}{\sin\beta \cdot \sin(\alpha+\gamma)} - \frac{(\mu \cdot \cos 2\alpha - 1)}{\sin 2\alpha} =$$

$$= (\cot(\angle ACB) + \cot(\angle BAC)) \cdot \frac{\sin\alpha \cdot \sin(\gamma+\beta)}{\sin\beta \cdot \sin(\alpha+\gamma)} - \frac{(\mu \cdot \cos 2\gamma - 1)}{\sin 2\gamma} \iff$$

$$\frac{(\cot(\angle ACB) + \cot(\angle BAC))}{\sin\beta \cdot \sin(\alpha+\gamma)} \cdot (\sin\alpha \cdot \sin(\gamma+\beta) - \sin\gamma \cdot \sin(\alpha+\beta)) =$$

$$= \frac{(\mu \cdot \cos 2\gamma - 1)}{\sin 2\gamma} - \frac{(\mu \cdot \cos 2\alpha - 1)}{\sin 2\alpha} \iff$$

$$(\cot(\angle ACB) + \cot(\angle BAC)) \cdot \sin 2\alpha \cdot \sin 2\gamma \cdot \left( \frac{\sin\alpha \cdot \sin(\gamma+\beta) - \sin\gamma \cdot \sin(\alpha+\beta)}{\sin\beta \cdot \sin(\alpha+\gamma)} \right) =$$

$$= \mu(\cos 2\gamma \cdot \sin 2\alpha - \cos 2\alpha \cdot \sin 2\gamma) - \sin 2\alpha + \sin 2\gamma \iff$$

$$(\cot(\angle ACB) + \cot(\angle BAC)) \cdot \sin 2\alpha \cdot \sin 2\gamma \cdot$$

$$\left( \frac{\sin\alpha \cdot \sin\beta \cdot \cos\gamma + \sin\alpha \cdot \sin\gamma \cdot \cos\beta - \sin\gamma \cdot \sin\alpha \cdot \cos\beta - \sin\gamma \cdot \sin\beta \cdot \cos\alpha}{\sin\beta \cdot \sin(\alpha+\gamma)} \right) =$$

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$$= \mu \cdot \sin(2\alpha - 2\gamma) - 2 \cdot \sin\left(\frac{2\alpha - 2\gamma}{2}\right) \cdot \cos\left(\frac{2\alpha + 2\gamma}{2}\right) \iff$$

$$(\cot(\angle ACB) + \cot(\angle BAC)) \cdot \sin 2\alpha \cdot \sin 2\gamma \cdot \frac{\sin\beta \cdot (\sin\alpha \cdot \cos\gamma - \sin\gamma \cdot \cos\alpha)}{\sin\beta \cdot \sin(\alpha+\gamma)} =$$

$$= \mu \cdot \sin(2\alpha - 2\gamma) - 2 \cdot \sin(\alpha - \gamma) \cdot \cos(\alpha + \gamma) \iff$$

$$(\cot(\angle ACB) + \cot(\angle BAC)) \cdot \sin 2\alpha \cdot \sin 2\gamma \cdot \frac{\sin(\alpha - \gamma)}{\sin(\alpha + \gamma)} =$$

$$= \mu \cdot 2\sin(\alpha - \gamma) \cdot \cos(\alpha - \gamma) - 2 \cdot \sin(\alpha - \gamma) \cdot \cos(\alpha + \gamma).$$

Since $\sin(\alpha - \gamma) \neq 0$ for $\alpha \neq \gamma$, we have that $\mu \cdot \cot\beta' = \mu \cdot \cot\phi \iff$

$$(\cot(\angle ACB) + \cot(\angle BAC)) \cdot \sin 2\alpha \cdot \sin 2\gamma =$$

$$= 2\mu \cdot \sin(\alpha + \gamma) \cdot \cos(\alpha - \gamma) - 2 \cdot \sin(\alpha + \gamma) \cdot \cos(\alpha + \gamma) \iff$$

$$(\cot(\angle ACB) + \cot(\angle BAC)) \cdot \sin 2\alpha \cdot \sin 2\gamma = 2\mu \cdot \sin(\alpha + \gamma) \cdot \cos(\alpha - \gamma) - \sin(2\alpha + 2\gamma)$$

$$\iff (\cot(\angle ACB) + \cot(\angle BAC)) \cdot \sin 2\alpha \cdot \sin 2\gamma =$$

$$= -\sin(2\alpha + 2\gamma) + 2\mu \cdot (\sin\alpha \cdot \cos\gamma + \sin\gamma \cdot \cos\alpha) \cdot (\cos\alpha \cdot \cos\gamma + \sin\alpha \cdot \sin\gamma) \iff$$

$$(\cot(\angle ACB) + \cot(\angle BAC)) \cdot \sin 2\alpha \cdot \sin 2\gamma =$$

$$= -\sin(2\alpha + 2\gamma) + 2\mu \cdot (\sin\alpha \cdot \cos\alpha \cdot (\sin^2\gamma + \cos^2\gamma) + \cos\gamma \cdot \sin\gamma \cdot (\sin^2\alpha + \cos^2\alpha).$$

Therefore $\mu \cdot \cot\beta' = \mu \cdot \cot\phi \iff$

$$(\cot(\angle ACB) + \cot(\angle BAC)) \cdot \sin 2\alpha \cdot \sin 2\gamma = -\sin(2\alpha + 2\gamma) + \mu \cdot (\sin 2\alpha + \sin 2\gamma),$$

which is true by (11). We then get $\beta' = \phi$ for $\beta', \phi \in (0, \pi)$.

Analogously we can conclude that $\alpha' = \theta$ and $\gamma' = \psi$ so that $ABC$ and $A'B'C'$ are reciprocal Jacobi triangles. □

# 3 Conclusions

Let $ABC$ an arbitray triangle and $A'B'C'$ its Jacobi triangle with angles $\alpha, \beta, \gamma$ constructed on the vertices $A, B, C$, respectively, as it is illustrated in Figure 1. We say that $ABC$ and $A'B'C'$ are reciprocal Jacobi triangles if $ABC$ can be obtained from $A'B'C'$ through a Jacobi construction, that is, if $\angle BA'C' = \angle CA'B'$, $\angle AB'C' = \angle CB'A'$ and $\angle AC'B' = \angle BC'A'$. It is possible to determine reciprocal Jacobi triangles by studying the following trigonometric condition

$$\frac{sin(\angle BAC + 2\alpha)}{sin(\angle BAC)} = \frac{sin(\angle ABC + 2\beta)}{sin(\angle ABC)} = \frac{sin(\angle ACB + 2\gamma)}{sin(\angle ACB)} \tag{12}$$

which involves only the internal angles of the triangle $ABC$ and $\alpha, \beta, \gamma$. In [2] it was shown that (12) is a necessary condition to $ABC$ and $A'B'C'$ be reciprocal Jacobi triangles. In this work we proved that (12) is also a sufficient condition, which was an open question since 2015.

# References

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