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Local well-posedness for 3D micropolar fluid system in Besov-Morrey spaces

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Resumo: *We show a local-in-time existence result for the 3D micropolar fluid system in the framework of Besov-Morrey spaces. The initial data class is larger than the previous ones and contains strongly singular functions and measures.*

Palavras-chave: *Micropolar fluids, well-posedness, Besov-Morrey spaces*

1 Introduction

In this work we are concerned with a system of equations describing a viscous incompressible homogeneous micropolar fluid filling the whole space $\mathbb{R}^3$ and with density equal to one. This system was introduced by A.C. Eringer in [1] and can be used to model the behavior of some fluids under micro-rotation effects caused by rigid suspended particles in a viscous medium. Examples of those are polymeric fluids, animal blood, liquid crystals, ferro-liquids, and many others. These fluids cannot be modeled by only using Navier-Stokes equations due to the important role played by their microstructures which make them to be non-newtonian fluids with nonsymmetric stress tensor.

The initial value problem (IVP) for the micropolar system reads as

$$\frac{\partial u}{\partial t} - (\chi+\nu)\Delta u + u\cdot\nabla u + \nabla\pi - 2\chi\nabla\times\omega = 0, \quad x\in\mathbb{R}^3, t>0, \tag{1.1}$$

$$\frac{\partial \omega}{\partial t} - \mu\Delta\omega + u\cdot\nabla\omega + 4\chi\omega - \kappa\nabla(\nabla\cdot\omega) - 2\chi\nabla\times u = 0, \quad x\in\mathbb{R}^3, t>0, \tag{1.2}$$

$$\nabla\cdot u = 0, \quad x\in\mathbb{R}^3, t>0, \tag{1.3}$$

$$u|_{t=0} = u_0,\ \nabla\cdot u_0 = 0 \text{ and } \omega|_{t=0} = \omega_0,\ x\in\mathbb{R}^3, \tag{1.4}$$

where the vector $u(x,t)$ is the linear velocity of the fluid, the scalar $\pi(x,t)$ represents the pressure, and $\omega(x,t)$ is the rotation velocity field of particles. The symbols $\nabla\cdot u$ and $\nabla\times u$ stand respectively for the divergence and rotational of the field u. The equations (1.1)-(1.3) are completed with Dirichlet conditions at infinity, that is, $u,\omega\to 0$ as $|x|\to\infty$. The fluid physical characteristics are determined by the constants ν,χ,κ,μ, where ν denotes the Newtonian viscosity and the parameters χ,κ,μ are viscosities related to the rotation field of particles ω (see [4]). The data u_0 and ω_0 are respectively the initial linear and rotation velocity. For simplicity of exposition, we assume $\chi=\nu=1/2$ and $\kappa=\mu=1$.

The micro-rotation influence on velocity field u disappears when $\chi = 0$ or $\omega = 0$ in (1.1)-(1.4), and then the 3D Navier–Stokes equations (3DNS) and their newtonian structure is recovered. We have a rich literature about existence of solutions for 3DNS in several frameworks, such as Lebesgue space L^p, weak$-L^p$ ($p \geq 3$), Morrey spaces $M_{p,\lambda}$ ($\lambda \in [0,3)$, $p \geq 3-\lambda$), PM^a ($2 \leq a < 3$), Besov spaces $B^{-k}_{p,\infty}$ ($0 < k \leq 1 - \frac{3}{p}$), BMO^{-1}_R ($0 < R \leq \infty$), Besov-Morrey spaces $N^{-s}_{p,\lambda,\infty}$ with $p \in [1,\infty)$, $0 \leq \lambda < 3$ and $0 < s \leq 1 - \frac{3-\lambda}{p}$, and some others. For $p = 1$, there is no an inclusion relation between BMO^{-1}_R and $N^{-s}_{p,\lambda,\infty}$ (see e.g. [5, p.18]) and they are maximal classes for local-in-time existence in the whole space $\mathbb{R}^3$.

It is natural to wonder which of those existence results for 3DNS could be extended for a fluid under the effect of micro-rotations and with a non-newtonian structure.

The goal of this work is to prove a local well-posedness result in a new setting whose initial data class is maximal for existence of solutions for (1.1)-(1.4). We consider the framework of Besov-Morrey spaces $N^{-s}_{p,\lambda,\infty}$ which contain strongly singular functions and measures supported in either points (Diracs), filaments or surfaces (see e.g. [2, Remark 3.3] for more details). This class is larger than the previous ones in view of the continuous inclusions

$$L^p \subset \text{weak-}L^p \subset B^{-k}_{q,\infty} \subset N^{-s}_{r,\lambda,\infty} \text{ and } PM^a \subset B^{-k}_{q,\infty} \subset N^{-s}_{r,\lambda,\infty}, \tag{1.5}$$

when $\frac{3}{p} = 3 - a = k + \frac{n}{q} = s + \frac{n-\lambda}{r}$ (in other words, the spaces in (1.5) have the same scaling).

2 Preliminaries

2.1 Function spaces and definitions

For $1 \leq p < \infty$ and $0 \leq \lambda < n$, the local Morrey space $M_{p,\lambda} = M_{p,\lambda}(\mathbb{R}^n)$ is defined as

$$M_{p,\lambda} = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{p,\lambda} < \infty \right\}, \tag{2.1}$$

where

$$\|f\|_{p,\lambda} = \sup_{x_0 \in \mathbb{R}^n,\ 0<R\leq 1} \left(R^{-\frac{\lambda}{p}} \|f\|_{L^p(B_R(x_0))} \right) \tag{2.2}$$

and $B_R(x_0) \subset \mathbb{R}^n$ is the open ball with center x and radius R. The space $M_{p,\lambda}$ endowed with $\|\cdot\|_{p,\lambda}$ is a Banach space. When $p = 1$, $M_{p,\lambda}$ should be understood as a space of Radon measures and the L^p-norm in (2.2) as the total variation of the measure f computed on $B_R(x_0)$.

Hölder inequality holds true in the framework of Morrey spaces. Precisely, if $1 \leq p_i \leq \infty$ and $0 \leq \lambda_i < n$ with $\dfrac{1}{p_3} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$ and $\dfrac{\lambda_3}{p_3} = \dfrac{\lambda_1}{p_1} + \dfrac{\lambda_2}{p_2}$, then

$$\|fg\|_{p_3,\lambda_3} \leq \|f\|_{p_1,\lambda_1} \|g\|_{p_2,\lambda_2}. \tag{2.3}$$

Recalling the notation $M^s_{p,\lambda} = (-\Delta)^{-s/2} M_{p,\lambda}$ for Sobolev Morrey spaces, the inhomogeneous Besov-Morrey space $N^s_{p,\lambda,q}$ is the following interpolation space

$$(M^{s_1}_{p,\lambda}, M^{s_2}_{p,\lambda})_{\theta,q} = N^s_{p,\lambda,q}, \tag{2.4}$$

where $\theta \in (0,1)$ and $s = (1-\theta)s_1 + \theta s_2$ with $s_1 \neq s_2$. In view of (2.4), the reiteration theorem implies that

$$(N^{s_1}_{p,\lambda,q_1}, N^{s_2}_{p,\lambda,q_2})_{\theta,q} = N^s_{p,\lambda,q}, \tag{2.5}$$

where $1 \leq q, q_1, q_2 \leq \infty$ with $q^{-1} = (1-\theta)q_1^{-1} + \theta q_2^{-1}$ and $\theta \in (0,1)$.

The space $N^s_{p,\lambda,q}$ can be characterized via norms based on dyadic decompositions. For that matter, let $D_0 = \{\xi \in \mathbb{R}^n : |\xi| < 1\}$ and let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be such that $\text{supp}(\widehat{\varphi}_0) \subset D_0$ and

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$\widehat{\varphi}_0(\xi) = 1$ if $|\xi| \leq \frac{2}{3}$. Define $\varphi_k = 2^{kn}\varphi_0(2^k\xi)$ and $\psi_k = \varphi_{k+1} - \varphi_k$ for all $k \in \{0\} \cup \mathbb{N}$. Then $\widehat{\varphi}_k(\xi) = \widehat{\varphi}_0(2^{-k}\xi)$, $\widehat{\psi}_k = \widehat{\varphi}_{k+1} - \widehat{\varphi}_k$, $\mathrm{supp}(\widehat{\psi}_k) \subset \{\xi \in \mathbb{R}^n : 2^{k-1} < |\xi| < 2^{k+1}\}$ and

$$\widehat{\varphi}_0 + \sum_{k=0}^{\infty} \widehat{\psi}_k(\xi) = 1, \text{ for all } \xi.$$

For $f \in \mathcal{S}'$, consider the quantity $\|f\|_{N^s_{p,\lambda,q}}$ given by

$$\begin{cases} \|\varphi_0 * f\|_{p,\lambda} + \left(\sum_{k=0}^{\infty}(2^{ks}\|\psi_k * f\|_{p,\lambda})^q\right)^{\frac{1}{q}}, \text{if } \ 1 \leq p \leq \infty, \ 1 \leq q < \infty, \ s \in \mathbb{R}. \\ \|\varphi_0 * f\|_{p,\lambda} + \sup_{k\in\{0\}\cup\mathbb{N}}(2^{ks}\|\psi_k * f\|_{p,\lambda}) \text{, if } \ 1 \leq p \leq \infty, \ q = \infty, \ s \in \mathbb{R}. \end{cases} \tag{2.6}$$

We have that

$$N^s_{p,\lambda,q} = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{N^s_{p,\lambda,q}} < \infty\} \tag{2.7}$$

and the pair $(N^s_{p,\lambda,q}, \|\cdot\|_{N^s_{p,\lambda,q}})$ is a Banach space. The inclusion $N^s_{p,\lambda,q_1} \subset N^s_{p,\lambda,q_2}$ is continuous for $1 \leq q_1 \leq q_2 \leq \infty$, and

$$N^0_{p,\lambda,1} \subset M_{p,\lambda} \subset N^0_{p,\lambda,\infty}, \tag{2.8}$$

for all $1 \leq p < \infty$ and $0 \leq \lambda < n$. Finally we recall the Sobolev type embedding

$$N^{s_2}_{p_2,\lambda,q_2} \subset N^{s_1}_{p_1,\lambda,q_1}, \quad \text{for } p_1 > p_2 \text{ and } s_1 - \frac{n-\lambda}{p_1} = s_2 - \frac{n-\lambda}{p_2}\ . \tag{2.9}$$

The following lemma can be found in [6] and gives estimates for some multiplier operators acting in $N^s_{p,\lambda,r}$.

Lemma 2.1. *Let $m, s \in \mathbb{R}$, $1 \leq p < \infty$, $0 \leq \lambda < n$ and $1 \leq r \leq \infty$. Let $P(\xi) \in C^{[n/2]+1}(\mathbb{R}^n)$ where $[\cdot]$ stands for the greatest integer function. Assume that there is $A > 0$ such that*

$$\left|\frac{\partial^{|\alpha|}P}{\partial\xi^\alpha}(\xi)\right| \leq A\,\langle\xi\rangle^{m-|\alpha|}, \ \textit{where } \ \langle\xi\rangle = (1+|\xi|^2)^{1/2},$$

for all $\xi \in \mathbb{R}^n$ and $|\alpha| \leq [n/2]+1$. Then the operator $P(D)$ is bounded from $N^s_{p,\lambda,r}$ to $N^{s-m}_{p,\lambda,r}$ and satisfies the estimate

$$\|P(D)u\|_{N^{s-m}_{p,\lambda,r}} \leq CA\|u\|_{N^s_{p,\lambda,r}},$$

where $C > 0$ is a constant depending only on s, m, p, λ.

2.2 Mild solutions

Recall that we are considering $\chi = \nu = 1/2$ and $\kappa = \mu = 1$ in (1.1)-(1.4). After applying the Leray projector in (1.1), we obtain the system

$$\begin{cases} \partial_t u - \Delta u - \nabla \times \omega + \mathbb{P}(u \cdot \nabla u) = 0 \\ \partial_t \omega - \Delta\omega + u \cdot \nabla\omega + 2\omega - \nabla(\nabla \cdot \omega) - \nabla \times u = 0 \\ u|_{t=0} = u_0, \ \nabla \cdot u_0 = 0, \ \text{and } \omega|_{t=0} = \omega_0 \end{cases}. \tag{2.10}$$

The linearized one associated to (2.10) is

$$\begin{cases} \partial_t u - \Delta u - \nabla \times \omega = 0 \\ \partial_t \omega - \Delta\omega + 2\omega - \nabla(\nabla \cdot \omega) - \nabla \times u = 0 \\ u|_{t=0} = u_0, \ \nabla \cdot u_0 = 0, \ \text{and } \omega|_{t=0} = \omega_0 \end{cases}. \tag{2.11}$$

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Proceeding as in [3], we apply the Fourier transform in (2.11) and use the notation $y = [u, \omega]$ in order to obtain

$$\begin{cases} \partial_t \hat{y} + A(\xi)\hat{y} = 0 \\ \hat{y}(\xi, 0) = (\hat{u}_0, \hat{\omega}_0) \end{cases}, \tag{2.12}$$

where

$$A(\xi) = \begin{pmatrix} |\xi|^2 I & B(\xi) \\ B(\xi) & R(\xi) + (|\xi|^2 + 2)I \end{pmatrix} = A_1(\xi) + A_2(\xi) + A_3(\xi), \tag{2.13}$$

with

$$A_1 = \begin{pmatrix} |\xi|^2 I & 0 \\ 0 & (|\xi|^2 + 2)I \end{pmatrix}, A_2 = \begin{pmatrix} 0 & B(\xi) \\ B(\xi) & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 0 & R(\xi) \end{pmatrix}, \tag{2.14}$$

and

$$R(\xi) = \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{pmatrix} \quad \text{and} \quad B(\xi) = i \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & -\xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}.$$

For each $t \geq 0$, we define the operator $G_A(t)$ via Fourier variables by

$$\widehat{G_A(t) y_0}(\xi) = e^{-A(\xi)t} \hat{y_0}, \tag{2.15}$$

where $A(\xi)$ has been defined in (2.13). From [3, inequality (20), p.1430] with $\gamma = \beta = 1$, we have the pointwise estimate

$$\left| e^{-tA(\xi)} \right| \leq C e^{-|\xi|^2 t}. \tag{2.16}$$

Notice that the family $\{G_A(t)\}_{t \geq 0}$ is formally a semigroup and $y = [u, \omega] = G_A(t) y_0$ is the solution of the linearized problem (2.11). Then, according to Duhamel's principle, the problem (2.10) is formally equivalent to the integral system

$$y(x,t) = G_A(t) y_0 - \int_0^t G_A(t-s) \mathbb{P} \nabla \cdot (u \otimes y) ds, \tag{2.17}$$

where

$$\mathbb{P} \nabla \cdot (u \otimes y) = [\mathbb{P} \nabla \cdot (u \otimes u), \nabla \cdot (u \otimes \omega)].$$

Throughout this paper, solutions of (2.17) are called mild ones for (1.1)-(1.4) (or for (2.10)).

3 Results

The purpose of this section is to state our existence result for the Cauchy problem (1.1)-(1.4). Given a vector space $X \subset \left(\mathcal{S}'(\mathbb{R}^3)\right)^3$, we denote X^σ as the set of all $u \in X$ such that $div(u) = 0$ in $\mathcal{S}'(\mathbb{R}^3)$.

Let $\eta, \beta > 0$, $1 \leq q < \infty$, $0 \leq \lambda < 3$ and $\eta_0 = \frac{1}{2} - \frac{3-\lambda}{4q}$. Many times, spaces of scalar and vector functions will be denoted abusively in the same way, e.g. $N^{-\beta}_{q,\lambda,\infty} = (N^{-\beta}_{q,\lambda,\infty})^3$. Also, $N^{-\beta,\sigma}_{q,\lambda,\infty}$ stands for $((N^{-\beta}_{q,\lambda,\infty})^3)^\sigma$.

We will look for local-in-time mild solutions $[u(x,t), \omega(x,t)]$ in the class $\mathcal{X}_T$ defined by

$$\left\{ [u, \omega] \in BC\left((0,T); N^{-\beta,\sigma}_{q,\lambda,\infty} \times N^{-\beta}_{q,\lambda,\infty}\right) : [t^\eta u, t^\eta \omega] \in BC\left((0,T); (M_{2q,\lambda})^2\right) \right\}, \tag{3.1}$$

which is a Banach space with norm

$$\|[u, \omega]\|_{\mathcal{X}_T} = \sup_{0<t<T} \|[u(\cdot,t), \omega(\cdot,t)]\|_{N^{-\beta}_{q,\lambda,\infty}} + \sup_{0<t<T} t^\eta \|[u(\cdot,t), \omega(\cdot,t)]\|_{2q,\lambda}, \tag{3.2}$$

where $\|[\cdot,\cdot]\|_{N^{-\beta}_{q,\lambda,\infty}} = \|[\cdot,\cdot]\|_{(N^{-\beta}_{q,\lambda,\infty})^2}$ and $\|[\cdot,\cdot]\|_{2q,\lambda} = \|[\cdot,\cdot]\|_{(M_{2q,\lambda})^2}$.

The initial data is taken in the class

$$[u_0,\omega_0] \in N^{-\beta,\sigma}_{q,\lambda,\infty} \times N^{-\beta}_{q,\lambda,\infty}. \tag{3.3}$$

We are ready to state our well-posedness result.

Theorem 3.1. *Assume that $1 \le q < \infty$, $0 \le \lambda < 3$, $0 < \beta < 1 - \frac{3-\lambda}{q}$ and $\frac{\beta}{2} + \frac{3-\lambda}{4q} \le \eta < \min\left\{\eta_0, \frac{\beta-1}{4} + \frac{1}{2}\right\}$. Suppose that the initial data $[u_0,\omega_0]$ belongs to the class (3.3).*

(i) (Existence and uniqueness) There exists $T > 0$ such that (1.1)-(1.4) has a mild solution $y = [u,\omega] \in \mathcal{X}_T$, which is the unique one in a suitable closed ball $\mathcal{B}_r$ of $\mathcal{X}_T$ whose radius $r > 0$ increases with the size of the initial data.

(ii) (Continuous dependence) The solution $[u,\omega]$ depends continuously on initial data $[u_0,\omega_0]$.

4 Proofs

In order to perform a contraction argument, we need to obtain estimates in Besov-Morrey spaces for the linear and bilinear terms of the integral equation (2.17). Let us start with core estimates for the semigroup $\{G_A(t)\}_{t\ge 0}$.

4.1 Estimates for $G_A(t)$ in $N^s_{p,\lambda,r}$-spaces

The next lemma gives estimates for $\{G_A(t)\}_{t\ge 0}$ on spaces $N^s_{p,\lambda,r}$. In particular, these operators are well defined in the setting of $N^s_{p,\lambda,r}$-spaces.

Lemma 4.1. *Let $s,\beta \in \mathbb{R}$, $s \le \beta$, $1 \le q \le \infty$, $1 \le r \le \infty$ and $0 \le \lambda < 3$. There exists a constant $C > 0$ such that*

$$\|G_A(t)y\|_{N^{\beta}_{q,\lambda,r}} \le C(1+t)^2(1+t^{\frac{s-\beta}{2}}) \quad \|y\|_{N^s_{q,\lambda,r}}, \tag{4.1}$$

for all $t > 0$ and $y \in N^s_{q,\lambda,r}$. Furthermore, if $s < \beta$, then

$$\|G_A(t)y\|_{N^{\beta}_{q,\lambda,1}} \le C(1+t)^2(1+t^{\frac{s-\beta}{2}}) \quad \|y\|_{N^s_{q,\lambda,\infty}}, \tag{4.2}$$

for all $t > 0$ and $y \in N^s_{q,\lambda,\infty}$.

4.2 Bilinear estimates

In the remainder of this work, we use the following notation for the bilinear operator appearing in (2.17)

$$B(y_1,y_2) = -\int_0^t G_A(t-s)\mathbb{P}\nabla\cdot(u_1\otimes y_2)ds, \tag{4.3}$$

where $y_1 = [u_1,\omega_1], y_2 = [u_2,\omega_2]$.

Lemma 4.2. *Let $0 < T < \infty$. Under the hypotheses of Theorem 3.1, there exist constants $K_1 = K_1(T), K_2 = K_2(T) > 0$ such that*

$$\sup_{0<t<T} \|B(y_1,y_2)\|_{N^{-\beta}_{q,\lambda,\infty}} \le K_1 \sup_{0<t<T} t^{\eta}\,\|y_1(\cdot,t)\|_{2q,\lambda} \sup_{0<t<T} t^{\eta}||y_2(\cdot,t)||_{2q,\lambda}, \tag{4.4}$$

$$\sup_{0<t<T} t^{\eta}\,\|B(y_1,y_2)\|_{2q,\lambda} \le K_2 \sup_{0<t<T} t^{\eta}\,\|y_1(\cdot,t)\|_{2q,\lambda} \sup_{0<t<T} t^{\eta}||y_2(\cdot,t)||_{2q,\lambda}, \tag{4.5}$$

for all $y_1,y_2 \in \mathcal{X}_T$. Moreover, $K_i(T) \to 0^+$ as $T \to 0^+$, for $i = 1,2$.

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4.3 Proof of Theorem 3.1.

Part (i): Recall the notation $B(\cdot,\cdot)$ in (4.3). Lemma 4.2 yields

$$\begin{aligned}
\|B(y,z)\|_{\mathcal{X}_T} &= \sup_{0<t<T} \|B(y,z)\|_{N^{-\beta}_{q,\lambda,\infty}} + \sup_{0<t<T} t^{\eta} \|B(y,z)\|_{2q,\lambda} \\
&\le (K_1(T)+K_2(T)) \left(\sup_{0<t<T} t^{\eta} \|y(\cdot,t)\|_{2q,\lambda} \sup_{0<t<T} t^{\eta} \|z(\cdot,t)\|_{2q,\lambda} \right) \\
&\le K(T) \|y\|_{\mathcal{X}_T} \|z\|_{\mathcal{X}_T}, \qquad (4.6)
\end{aligned}$$

where $K(T) = K_1(T) + K_2(T)$.

Let $\tau_{q,\lambda} = \frac{3-\lambda}{q}$. Using the inclusion $N^0_{q,\lambda,1} \hookrightarrow M_{q,\lambda}$ (see (2.8)), Sobolev type embedding (2.9), and Lemma 4.1, we obtain

$$\begin{aligned}
\|G_A(t)y_0\|_{\mathcal{X}_T} &= \sup_{0<t<T} \|G_A(t)y_0\|_{N^{-\beta}_{q,\lambda,\infty}} + \sup_{0<t<T} t^{\eta} \|G_A(t)y_0\|_{2q,\lambda} \\
&\le 2C(1+T)^2 \sup_{0<t<T} \|y_0\|_{N^{-\beta}_{q,\lambda,\infty}} + C \sup_{0<t<T} t^{\eta} \|G_A(t)y_0\|_{N^0_{2q,\lambda,1}} \\
&\le 2C(1+T)^2 \sup_{0<t<T} \|y_0\|_{N^{-\beta}_{q,\lambda,\infty}} + C \sup_{0<t<T} t^{\eta} \|G_A(t)y_0\|_{N^{\frac{\tau_{q,\lambda}}{2}}_{q,\lambda,1}} \\
&\le 2C(1+T)^2 \sup_{0<t<T} \|y_0\|_{N^{-\beta}_{q,\lambda,\infty}} \\
&\quad + C \sup_{0<t<T} t^{\eta}(1+t)^2 (1+t^{-\frac{\tau_{q,\lambda}}{4}-\frac{\beta}{2}}) \|y_0\|_{N^{-\beta}_{q,\lambda,\infty}} \\
&\le C(1+T)^2 \left(2 + \left(T^{\eta} + \ T^{\eta-\frac{\tau_{q,\lambda}}{4}-\frac{\beta}{2}}\right)\right) \|y_0\|_{N^{-\beta}_{q,\lambda,\infty}} \\
&= C_T \|y_0\|_{N^{-\beta}_{q,\lambda,\infty}}, \qquad (4.7)
\end{aligned}$$

because $\eta \ge \frac{\beta}{2} + \frac{\tau_{q,\lambda}}{4}$. Consider the map Φ defined by

$$\Phi(y) = G_A(t)y_0 + B(y,y). \qquad (4.8)$$

Let $T > 0$ and $\mathcal{B}_r = \{y \in \mathcal{X}_T; \|u\|_{\mathcal{X}_T} \le 2r\}$ where

$$r = C_T \|y_0\|_{N^{-\beta}_{q,\lambda,\infty}}, \qquad (4.9)$$

and C_T is as in (4.7). Since $K_i(T) \to 0^+$ as $T \to 0^+$ and $K(T) = K_1(T) + K_2(T)$, we can choose $T > 0$ such that

$$\begin{aligned}
4K(T)r &= 4K(T)C_T \|y_0\|_{N^{-\beta}_{q,\lambda,\infty}} \\
&= C \|y_0\|_{N^{-\beta}_{q,\lambda,\infty}} 4(1+T)^2 \left(2 + \left(T^{\eta} + \ T^{\eta-\frac{\tau_{q,\lambda}}{4}-\frac{\beta}{2}}\right)\right) K(T) < 1.
\end{aligned}$$

It follows from bilinearity and (4.6) that

$$\begin{aligned}
\|\Phi(y) - \Phi(z)\|_{\mathcal{X}_T} &= \|B(y,y) - B(z,z)\|_{\mathcal{X}_T} \\
&\le K\|y-z\|_{\mathcal{X}_T} (\|y\|_{\mathcal{X}_T} + \|z\|_{\mathcal{X}_T}) \qquad (4.10) \\
&\le 4rK(T)\|y-z\|_{\mathcal{X}_T}, \qquad (4.11)
\end{aligned}$$

for all $y, z \in \mathcal{B}_r$. Also, using the inequality (4.10) with $z = 0$, we get

$$\begin{aligned}
\|\Phi(y)\|_{\mathcal{X}_T} &\le \|\Phi(0)\|_{\mathcal{X}_T} + \|\Phi(y) - \Phi(0)\|_{\mathcal{X}_T} \\
&\le \|G_A(t)u_0\|_{\mathcal{X}_T} + K\|y\|^2_{\mathcal{X}_T} \\
&\le r + 4r^2K(T) \le 2r, \ \text{for } y \in \mathcal{B}_r, \qquad (4.12)
\end{aligned}$$

because $4K(T)r < 1$. The estimates (4.11) and (4.12) show that $\Phi : \mathcal{B}_r \to \mathcal{B}_r$ is a contraction and has a unique fixed point in $\mathcal{B}_r$. This one is the unique solution $y(x,t)$ for the integral equation (2.17) satisfying $\|y\|_{\mathcal{X}_T} \leq 2r$.

Part (ii): Let y_1, y_2 be two solutions obtained in *item (i)* with respective data $y_{0,1}, y_{0,2}$ and existence times T_1, T_2. We have that $\|y_1\|_{\mathcal{X}_{T_1}} \leq 2r_1$ and $\|y_2\|_{\mathcal{X}_{T_2}} \leq 2r_2$, where $r_i = C_{T_i} \|y_i\|_{N^{-\beta}_{q,\lambda,\infty}}$ (see (4.9)) and $0 < 4K(T_i)r_i < 1$. Since $K(T)$ decreases with $T > 0$, we have that $0 < 4K(T)r_i < 1$, for $i = 1, 2$, and $T = \min\{T_1, T_2\}$. Thus, taking $r = \max\{r_1, r_2\}$, note that $0 < 4K(T)r < 1$.

Now, using (4.11), it follows that

$$\begin{aligned}
\|y_1 - y_2\|_{\mathcal{X}_T} &= \|G_A(t)y_{0,1} - G_A(t)y_{0,2} + B(y_1, y_1) - B(y_2, y_2)\|_{\mathcal{X}_T} \\
&\leq \|G_A(t)(y_{0,1} - y_{0,2})\|_{\mathcal{X}_T} + K(T)\|y_1 - y_2\|_{\mathcal{X}_T}(\|y_1\|_{\mathcal{X}_T} + \|y_2\|_{\mathcal{X}_T}) \\
&\leq C_T \|y_{0,1} - y_{0,2}\|_{N^{-\beta}_{q,\lambda,\infty}} + 2(r_1 + r_2)K(T)\|y_1 - y_2\|_{\mathcal{X}_T} \\
&\leq C_T \|y_{0,1} - y_{0,2}\|_{N^{-\beta}_{q,\lambda,\infty}} + 4rK(T)\|y_1 - y_2\|_{\mathcal{X}_T},
\end{aligned}$$

and then

$$\|y_1 - y_2\|_{\mathcal{X}_T} \leq \frac{C_T}{1 - 4rK(T)} \|y_{0,1} - y_{0,2}\|_{N^{-\beta}_{q,\lambda,\infty}},$$

which implies the desired continuity of the data-solution map. □

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