

Ergodicity of a 2-to-1 baker map

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Abstract. We encode a 2-to-1 baker map using the so called zip shift map, defined on an extended two-sided symbolic space. Moreover show that the 2-to-1 baker map is a $(2, 4)$ -Bernoulli transformation. As an application, we show that it is strongly mixing and ergodic.

Keywords. Ergodic Theory, Deterministic Chaos, Bernoulli Transformations, Zip shift map.

1 Introduction

The transformation, known as the *baker's map* is topologically conjugated with the invertible Smale horseshoe map [5] and preserves the two-dimensional Lebesgue measure. It provides a model of deterministic chaos [1] in an invertible context, that is rich in ergodic properties, such as strong mixing and ergodicity. Many dynamical systems are known to be equivalent to the baker's transformation. This process which is known as the Bernoulli Process is similar to the coin toss. In ergodic theory, such transformations are called Bernoulli transformations [4].

In this short paper, we study a 2-to-1 baker's transformation, which represents a non-invertible model of deterministic chaos. In [2], the authors introduced some extended two-sided shift map, called the zip shift map, and the corresponded shift space called the zip shift space. In [3] the (m, l) -Bernoulli transformations (finite-to-1) are introduced, which are the prototype version of the Bernoulli transformations, that are measure theoretically conjugated with a zip shift map instead of a shift homeomorphism. With the help of these notions, we encode the 2-to-1 baker map and show that it is a $(2, 4)$ -Bernoulli transformation. Consequently, it is strongly mixing and an ergodic transformation.

2 An Extended Two-Sided Shift Map

The zip shift space and the zip shift maps are defined in [2]. The zip shift map is a local homeomorphism which is an extension of the known shift homeomorphism.

Let $Z = \{a_1, a_2, \dots, a_m\}$ and $S = \{0, 1, \dots, l - 1\}$ be two collection of symbols with $m \leq l$ and $\kappa : S \rightarrow Z$ a surjective factor map. Consider

$$\Sigma_Z := \prod_{-\infty}^{-1} Z \text{ and } \Sigma_S := \prod_0^{\infty} S, \quad \Sigma_{Z,S} := \Sigma_Z \times \Sigma_S \quad (1)$$

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Every $x \in \Sigma_{Z,S}$ can be seen in the form of $x = (\dots x_{-n} \dots x_{-1} \cdot x_0 x_1 \dots x_n \dots)$, such that $x_i \in Z$ if $i < 0$ and $x_i \in S$ if $i \geq 0$.

Define $\bar{d} : \Sigma_{Z,S} \times \Sigma_{Z,S} \rightarrow \mathbb{R}$ as,

$$\bar{d}(x, y) = \begin{cases} \frac{1}{2^{M(x,y)}}, & \text{se } x \neq y \\ 0, & \text{se } x = y, \end{cases} \quad (2)$$

where $M(x, y) = \min\{|x_i|; x_i \neq y_i\}$. Then (Σ, \bar{d}) is a metric space and induces a topology, on Σ .

Definition 2.1 (Zip shift Space and Zip shift Map). Consider the $\Sigma_{Z,S}$ and the factor map $\kappa : S \rightarrow Z$. then the **Zip shift map** is defined as, $\sigma_\kappa : \Sigma_{Z,S} \rightarrow \Sigma_{Z,S}$ such that,

$$\sigma_\kappa(\dots x_{-1} \cdot x_0 x_1 \dots) = (\dots x_{-1} \kappa(x_0) \cdot x_1 \dots). \quad (3)$$

The pair $(\Sigma_{Z,S}, \sigma_\kappa)$ is called a **Zip shift Space**. By an (m, l) -zip shift map we mean the zip shift map defined on a zip shift space with (m, l) -symbols.

Example 2.1 Let $S = \{0, 1, 2\}$ and $Z = \{a\}$. With the corresponded factor map $\kappa : S \rightarrow Z$ defined as $\kappa(0) = \kappa(1) = \kappa(2) = a$ (constant map). Let $\Sigma = \Sigma_1^- \times \Sigma_3^+$, with $\Sigma_1^- = \prod_{-\infty}^{-1} \{a\}$ and $\Sigma_3^+ = \prod_0^{+\infty} \{0, 1, 2\}$. Then $\sigma_\kappa : \Sigma \rightarrow \Sigma$ is a zip shift map (Figure 1). ■

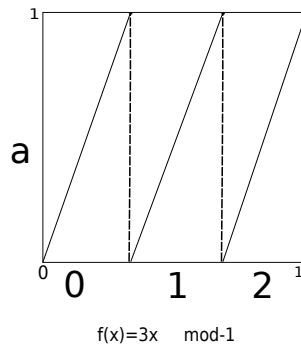


Figura 1: A 3-to-1 zip shift map.

Example 2.2 Let $S = \{0, 1, 2, 3\}$ and $Z = \{a, b\}$ where the corresponded factor map $\kappa : S \rightarrow Z$ is defined as $\kappa(0) = \kappa(2) = a$ and $\kappa(1) = \kappa(3) = b$. Let $\Sigma = \Sigma_2^- \times \Sigma_4^+$, with $\Sigma_2^- = \prod_{-\infty}^{-1} \{a, b\}$ and $\Sigma_4^+ = \prod_0^{+\infty} \{0, 1, 2, 3\}$. Then $\sigma_\kappa : \Sigma \rightarrow \Sigma$ is a zip shift map. Let $(t_n) = (\dots a b a b b \cdot 1 0 1' 1 1 0' \dots)$. One can verify that

$$\sigma_\kappa((\dots a b a b b \cdot 1 0 1' 1 1 0' \dots)) = (\dots a b a b b b \cdot 0 1' 1 1 0' \dots), \quad (4)$$

and

$$\sigma_\kappa^2((\dots a b a b b \cdot 1 0 1' 1 1 0' \dots)) = (\dots a b a b b b a \cdot 1' 1 1 0' \dots). \quad (5)$$

We will show that this zip shift map is isomorphic (mod-0) with a 2-to-1 baker map represented in Figure 2. ■

We equip the Σ with a σ -algebra and turn it into a probability measure space. Let define the basic cylinder sets as follows.

$$C_i^{s_i} = \{(t_n) \in \Sigma \mid t_i = s_i, i \in \mathbb{Z}, s_i \in Z, \text{ if } i < 0, \text{ and } s_i \in S, \text{ if } i \geq 0\}. \quad (6)$$

The $C_i^{s_i}$ presents the set of all sequences, that have s_i in the i -th entry. For $i, n \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$, one can define a cylinder set as follows.

$$C_i^{s_i, \dots, s_{i+k}} = \{(t_n) \in \Sigma : t_i = s_i, \dots, t_{i+k} = s_{i+k}\} = C_i^{s_i} \cap C_{i+1}^{s_{i+1}} \dots \cap C_{i+k}^{s_{i+k}}. \tag{7}$$

Lemma 2.1. *The collection of all cylinder sets of $\Sigma_{Z,S}$ is a topological basis for $(\Sigma_{Z,S}, \bar{d})$.*

Let \mathcal{C} denotes the smallest σ -algebra (Borel σ -algebra) generated by the family of all cylinder sets. In order to change the (Σ, \mathcal{C}) into a probability measure space, one considers $P = (p_0, \dots, p_{l-1})$ as a probability distribution on S and defines the measure of the basic cylinder sets as following.

$$\mu(C_i^{s_i}) = p_i, \quad i \geq 0 \text{ and, } \sum_{i=0}^{l-1} p_i = 1. \tag{8}$$

Observe that, considering the factor map $\kappa : S \rightarrow Z$, for $i < 0$, $\mu(C_i^{s_i}) = kp_i$, with, $k = \#(\kappa^{-1}(s_i))$ and, $\sum_{i=1}^m p_i = 1$. Then,

$$\begin{aligned} \mu(C_i^{s_i, \dots, s_{i+k}}) &= \mu(\{(t_n) \in \Sigma : t_i = s_i, \dots, t_{i+k} = s_{i+k}\}) \\ &= \mu(C_i^{s_i} \cap C_{i+1}^{s_{i+1}} \dots \cap C_{i+k}^{s_{i+k}}) \\ &= p_i \dots \cdot p_{i+k} \end{aligned} \tag{9}$$

and $(\Sigma, \mathcal{C}, \mu)$ is a probability measure space. The following Propositions are from [3].

Proposition 2.1. *Let $P = (p_0, \dots, p_{l-1})$ be a probability distribution for the elements of the set $S = \{0, 1, \dots, l-1\}$ and $(\Sigma, \mathcal{C}, \mu)$ a measure space defined on $\Sigma = \Sigma_{Z,S}$ as above. Then $\sigma_\kappa : \Sigma \rightarrow \Sigma$ preserves the probability measure μ .*

Proposition 2.2. *Let $\sigma_\kappa : \Sigma \rightarrow \Sigma$ be the zip shift map. Then, $\sigma_\kappa(\Sigma) = \Sigma$ and σ_κ is a local homeomorphism.*

Proposition 2.3. *The zip shift map $\sigma_\kappa : \Sigma \rightarrow \Sigma$ is strongly mixing and ergodic.*

Definition 2.2 ((m, l)-Bernoulli Transformations). *The locally invertible measure preserving transformation $T : X \rightarrow X$ defined on a Lebesgue space (X, \mathcal{B}, μ) have (m, l)-Bernoulli property or is a (m, l)-Bernoulli transformation if it is isomorphic (mod 0) with an (m, l)-zip shift map.*

Corollary 2.1. *As measure theoretical conjugacy preserves the strong mixing property, the (m, l)-Bernoulli transformations are strongly mixing and ergodic.*

3 The 2-to-1 baker map

Let $T : X \rightarrow X$, represents a 2-to-1 baker map with $X = [0, 1] \times [0, 1]$.

$$T(x, y) = \begin{cases} (4x, \frac{1}{2}y) & 0 \leq x < \frac{1}{4}, 0 \leq y \leq 1 \\ (4x - 1, \frac{1}{2}y + \frac{1}{2}) & \frac{1}{4} \leq x < \frac{1}{2}, 0 \leq y \leq 1 \\ (4x - 2, \frac{1}{2}y) & \frac{1}{2} \leq x < \frac{3}{4}, 0 \leq y \leq 1 \\ (4x - 3, \frac{1}{2}y + \frac{1}{2}) & \frac{3}{4} \leq x \leq 1, 0 \leq y \leq 1. \end{cases} \tag{10}$$

This transformation preserves the Lebesgue measure. Let $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ denotes the algebra made of the sets $[x, 0] \times [y, 0]$ on the unit square $X = [0, 1]^2$, where \mathcal{C}_1 is the algebra made of sets $[0, x]$ on $[0, 1]_x$ and \mathcal{C}_2 the algebra made of sets $[0, y]$ on $[0, 1]_y$. Set $\mathcal{B}(\mathcal{C})$ as the Borel σ -algebra generated by \mathcal{C} . Then using the Extension Theorem of Hahn and Kolmogorov [4], the two-dimensional Lebesgue

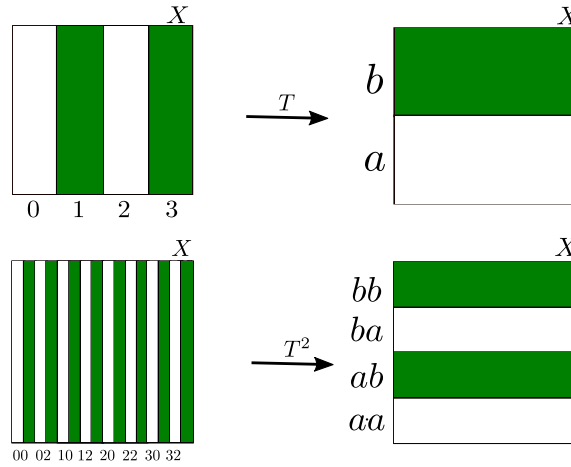


Figura 2: A 2-to-1 baker map.

measure $m : \mathcal{B}(\mathcal{C}) \rightarrow [0, 1]$ is uniquely defined. Let $(X, \overline{\mathcal{B}(\mathcal{C})}, m)$ denote the Lebesgue probability space defined on X . Then for any $B \in \mathcal{C}$, the $T_*m(B) = m(B)$, where $T_*m(B) = m(T^{-1}B)$. In fact, For $x \in [0, 1]$ and $y \in [0, \frac{1}{2}]$,

$$T^{-1}([0, x] \times [0, y]) = [0, \frac{x}{4}] \times [0, 2y] \cup [\frac{1}{2}, \frac{1}{2} + \frac{x}{4}] \times [0, 2y].$$

Therefore, $m(T^{-1}([0, x] \times [0, y])) = m([0, x] \times [0, y])$. And for $x \in [0, 1]$ and $y \in [\frac{1}{2}, 1]$,

$$T^{-1}([0, x] \times [1/2, y]) = [\frac{1}{4}, \frac{1}{4} + \frac{x}{4}] \times [0, 2y - 1] \cup [\frac{3}{4}, \frac{3}{4} + \frac{x}{4}] \times [0, 2y - 1].$$

Therefore, $m(T^{-1}([0, x] \times [1/2, y])) = m([0, x] \times [1/2, y])$. This shows that the Lebesgue measure is T -invariant.

Theorem 3.1. *The 2-to-1 baker map is a (2, 4)-Bernoulli Transformation.*

Proof. The 2-to-1 baker map is a transformation isomorphic (mod 0) with a (2, 4)-zip shift map defined on $\Sigma_{Z,S}$. Where, $Z = \{a, b\}$, $S = \{0, 1, 2, 3\}$. The factor code map $\kappa : S \rightarrow Z$ is defined as in Example 2.2. We give some numeric representation to a, b in which we can see $a = a_0$ and $b = b_1$. Let define the measurable transformation $\rho : \Sigma_{Z,S} \rightarrow X$ as follows.

$$\rho((s_i)) = (\sum_{i=1}^{\infty} \frac{s_{i-1}}{4^i} \pmod{1}, \sum_{i=1}^{\infty} \frac{s_{-i}}{2^i} \pmod{1}). \tag{11}$$

The map ρ corresponds any (s_i) almost uniquely to a point $s = (x, y) \in X$. It turns out that the forward and backward orbits of some points have more than one representation, we represent them by ∂ . Let $X_{\partial} = \bigcup_{-\infty}^{\infty} T^n(\partial)$ and $\Sigma_{\partial} = \rho^{-1}(X_{\partial})$. It is not difficult to verify that $\sigma_{\kappa}(\Sigma_{\partial}) = \Sigma_{\partial}$ and $m(X_{\partial}) = 0$. Let $\Sigma_0 = (\Sigma \setminus \Sigma_{\partial})$ and $X_0 = X \setminus X_{\partial}$. Consider the Lebesgue space $(X, \overline{\mathcal{B}}, m)$ with the Borel σ -algebra generated by subsets of the form $[0, x] \times [0, y]$. Then the 2-to-1 baker transformation preserves the Lebesgue measure. Moreover $\mu(\Sigma_0) = m(X_0) = 1$ and $\rho : \Sigma_0 \rightarrow X_0$ is a measure-preserving isomorphism. Observe that the push forward (use ρ^{-1}) of the Lebesgue measure defined on X_0 induces a probability measure μ on Σ_0 . By Proposition 2.3, σ_{κ} preserves the probability measure μ . Furthermore, $\rho \circ \sigma_{\kappa} = T \circ \rho$.

To show that, one rewrites the 2-to-1 baker map in the form of,

$$T(x, y) = (4x \pmod 1, \frac{1}{2}(y + \kappa(s_0))). \tag{12}$$

Note that $k(s_0)$ is considered by numeric representations of $a, b \in Z$. Then,

$$\begin{aligned} T(\rho((s_i))) &= T\left(\sum_{i=1}^{\infty} \frac{s_{i-1}}{4^i} \pmod 1, \sum_{i=1}^{\infty} \frac{s_{-i}}{2^i} \pmod 1\right) \\ &= \left(4 \sum_{i=1}^{\infty} \frac{s_{i-1}}{4^i} \pmod 1, \frac{\kappa(s_0)}{2} + \sum_{i=1}^{\infty} \frac{s_{-i}}{2^{i+1}} \pmod 1\right) \\ &= \left(\sum_{i=1}^{\infty} \frac{s_i}{4^i} \pmod 1, \sum_{i=1}^{\infty} \frac{s_{-i+1}}{2^i} \pmod 1\right) = \rho(\sigma_{\kappa}(s_i)), \end{aligned} \tag{13}$$

where, by statement 11,

$$\rho(\sigma_{\kappa}((s_i))) = \rho((s_{i+1})) = \left(\sum_{i=1}^{\infty} \frac{s_i}{4^i} \pmod 1, \sum_{i=1}^{\infty} \frac{s_{-i+1}}{2^i} \pmod 1\right). \tag{14}$$

Thus the 2-to-1 baker map is a (2, 4)–Bernoulli transformation. ■

Theorem 3.2. *The 2-to-1 baker map is an ergodic Transformation.*

Proof. The proof of this theorem arises from Theorem 3.1 and Corollary 2.1. ■

4 Conclusion

We showed that the 2-to-1 baker map is an (m, l) -Bernoulli transformation, i.e., we encoded it using an extended symbolic dynamic. As an application, we concluded that the 2-to-1 baker map is a conservative ergodic transformation.

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