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Grahps of stable plane-Gauss maps

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Abstract: In this paper, we present immersions of a given closed and oriented surface M in $\mathbb{R}^3$, where the two applications are stable: projection on the plane xy and a Gauss map . The projection on the plane can be seen as a stable map of surfaces on the plane.

keywords: Closed sufarces, Graphs, stable maps

1 Introduction

The singular set of stable maps from closed surfaces M to the plane consists of curves of double points, possibly containing isolated cusp points [4]. The apparent contour (i.e. the image of the singular set) is the number of immersed curves in $\mathbb{R}^2$ (possibly with cusps) whose self-intersections are all transverse and disjoint from the cusps. Graphs associated with the global study of stable applications of closed surfaces and oriented in the plane, was introduced in [6]. The singularities of a stable Gauss map, in Whitney's sense, being fold curves with isolated cusp points, are called the *parabolic set* of the surface ([1, 2, 9]). Each parabolic curve in this set separates a hyperbolic region from an elliptic region of the surface. Graphs associated with the global study of stable Gauss applications of closed surfaces and oriented in the plane, was introduced in [7].

The purpose of this work is to present immersion constructions of closed and oriented surfaces in 3-space, with the coordinate system (x, y, z), where the two maps are stable: projection on the plane xy, which can be seen as maps of surfaces in the plane, and the Gauss map. We show that any pair of trees can be realized as a graph of a pair of maps, one of them is a Gauss map and the other a map of the sphere in the plane.

2 Stable maps

Let M and N be smooth connected closed orientable surfaces and f, $g : M \to N$ be smooth maps between them. It is said that f is $\mathcal{A}$-*equivalent* (or *equivalent*) to g if there are orientation-preserving diffeomorphisms, $k : M \to M$ and $l : N \to N$, such that $g \circ k = l \circ f$. A smooth map $f : M \to N$ is said to be *stable* if all maps sufficiently close to f, in the Whitney C^∞-topology (see [4]), are equivalent to f.

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The concept of stability for a Gauss map of a surface immersed in $\mathbb{R}^3$ is slightly different from the general case of maps between surfaces in the sense that it depends on perturbations of the immersion rather than on those of the map itself.

Let $j : M \to \mathbb{R}^3$ be an immersion of M in $\mathbb{R}^3$. Consider the following maps:

1. A stable projection $p_v : j(M) \to \mathcal{P} \subset \mathbb{R}^3$, where $\mathcal{P}$ is a orthogonal plane to $v \in \mathbb{R}^3$ and p_v is the restriction to $j(M)$, of the projection of $\mathbb{R}^3$ in $\mathcal{P}$. A diffeomorphism $g : \mathcal{P} \to \mathbb{R}^2$ and $f_2 : M \to \mathbb{R}^2$ given by $f_2 = g \circ p_v \circ j$.

2. Gauss map $\mathcal{N}_j : j(M) \to \mathbb{S}^2$ of $j(M)$ and $f_3 : M \to \mathbb{S}^2$ given by $f_3 = \mathcal{N}_j \circ j$.

In this case, the map f_3 is called *Gauss map of M*, associated to the immersion j. The Gauss map $\mathcal{N}_j$ is said to be *stable* if there exists a neighborhood $\mathcal{U}_j$ of j in the space $\mathcal{I}(M, \mathbb{R}^3)$ of immersions of M into $\mathbb{R}^3$ such that for all $k \in \mathcal{U}_j$, the Gauss map $\mathcal{N}_k$ associated to k is $\mathcal{A}$-equivalent to $\mathcal{N}_j$.

Definition 2.1. *We say that the smooth map $f = (f_2, f_3) : M \to \mathbb{R}^2 \times \mathbb{S}^2$ is a stable plane-Gauss map if each f_i, $i = 2, 3$, is a stable map.*

Let $f = (f_2, f_3) : M \to \mathbb{R}^2 \times \mathbb{S}^2$ be a stable plane-Gauss map. A point of the surface M is a *regular* point of f_i, $(i = 2, 3)$, if the map f_i is a local diffeomorphism around that point and otherwise *singular*. We denote by Σf_i the *singular set* of f_i and its image $Bf_i = f_i(\Sigma f_i)$ is the *branch set* of f_i. We observe that regular points of f_3 corresponds, geometrically, to elliptic or hyperbolic points of $j(M)$. Singular points of f_3 corresponds to parabolic points of $j(M)$ (see [7]). By Whitney's Theorem (see [4]), Σf_i consists of closed curves with fold points, possibly containing isolated cusp points. Then the branch set of f_i, consists of a collection of closed curves immersed in the target surface with possible isolated cusps and self-intersections (double points). The orientation of the branch set is as follows: transverse a branch curve following the orientation with nearby points on our left that have two more inverse images than those on our right. The non-singular set (which is immersed in the target surface by the map f_i) consists of a finite number of regions.

3   Graph of stable plane-Gauss maps

The singular sets of two equivalent maps are equivalent, in the sense that, there is a diffeomorphism carrying one singular set onto the other and similarly for the branch sets. Thus any invariant diffeomorphism of singular sets or branch sets will automatically be a topological invariant of the map. Both the number of connected components of the singular set and the topological types of the regions are topological invariants. This information may be encoded on a weighted graph $\mathcal{G}^i(V^i, E^i, W^i)$, where the pair $(M, \Sigma f_i)$ may be reconstructed (up to diffeomorphism) (see [6, 7]). On the weighted graph $\mathcal{G}^i(V^i, E^i, W^i)$ defined by a stable map f_i each of its E^i *edges* corresponds to a path-component of the singular set of M and each of its V^i *vertices* to a different regular region of the surface. An edge is *incident* to a vertex if and only if the corresponding singular curve to the edge lies in the boundary of the regular region corresponding to the vertex (see Figure 1). A *weight* is defined as the genus of the region that represents and it is attached to each vertex. The number W^i is the sum of weights of all graph vertices. It will be called *total weight* of the graph. When $W^i = 0$, we will denote the graph simply by $\mathcal{G}^i(V^i, E^i)$ $(i = 2, 3)$.

It is remarkable that V^i represents the number of connected components of $M \setminus \Sigma f_i$; E^i the number of connected components of Σf_i and W^i the total sum of the genus of the components of $M \setminus \Sigma f_i$. Given orientations of the surfaces M, $\mathbb{R}^2$ and $\mathbb{S}^2$, a region of $M \setminus \Sigma f_2$ is *positive* if the map preserves orientation and *negative* otherwise. A region of $M \setminus \Sigma f_3$ is *positive* if it has positive Gaussian curvature and *negative* otherwise. A vertex of $\mathcal{G}^i(V^i, E^i, W^i)$ has a *positive* (or *negative*) label depending on whether the region that it represents is positive (or negative). Since

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each component of Σf_i is the boundary of a positive and a negative region, the signs of the vertices are assigned alternatively, hence the graph associated to stable map f_i is bipartite. We denote by $V^{i\pm}$ the number of positive (negative) vertices and $W^{i\pm}$ the total weight associated to the positive (negative) vertices.

Figure 1 illustrates an embedding of the torus in the 3-space, where f_2 is a map of the torus in the plane and f_3 is a Gauss map. Let us remark that the sets of singular curves of these maps f_2 and f_3 are not equivalent.
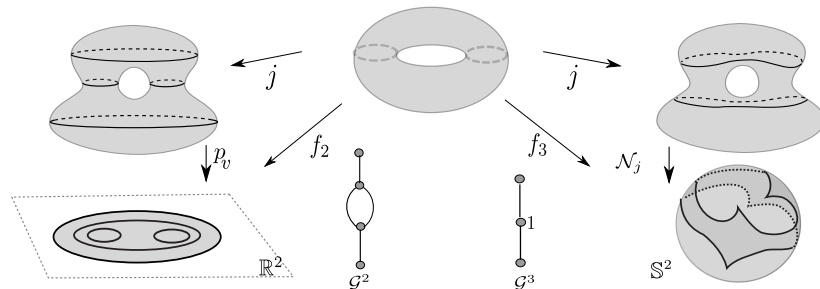

Figure 1: Example of stable plane-Gauss maps.

Definition 3.1. *Let $\mathcal{G}$ be a connected graph with non-negative integer weights on its vertices. We say that $\mathcal{G}$ is a (see Figure 1):*

1. *Mendes-graph or simply $\mathcal{M}$-graph if exists a smooth connected closed surface M, a smooth surface N and a stable map $f_2 : M \to N$ such that $\mathcal{G}$ is the graph of f_2 (in the sense of [6]).*

2. *Fuster-graph or simply $\mathcal{F}$-graph if exists a smooth connected closed surface M and an immersion $j : M \to \mathbb{R}^3$ whose stable Gauss map $f_3 : M \to \mathbb{S}^2$, associated to j, has $\mathcal{G}$ as its associated graph (in the sense of [7]).*

The Figure 1 illustrates the stable maps $f_2 : \mathbb{S}^2 \to \mathbb{R}^2$, whose $\mathcal{M}$-graph type $\mathcal{G}_1^2(4, 6, 0)$, and $f_3 : \mathbb{S}^2 \to \mathbb{S}^2$, whose $\mathcal{F}$-graph type $\mathcal{G}_2^3(3, 2, 1)$.

3.1 Graph of stable plane-Gauss maps

The graph $\mathcal{G}^i(V^i, E^i, W^i)$ is a global invariant and it classifies completely the topology of the stable maps regular set of f_i. Moreover, it will be used as a tool for the construction of examples of stable maps between surfaces.

The maps illustrates $f_2, g_2 : \mathbb{S}^2 \to \mathbb{R}^2$ in Figure 2 are non equivalent, where $j_i's$ indicates the respective embedding from $\mathbb{S}^2$ in $\mathbb{R}^3$ and p_v, $v = e_3$, the canonical projection. This example shows that the $\mathcal{M}$-graph is an invariants, because the apparent contour sets of f_2 and g_2 can not differ.
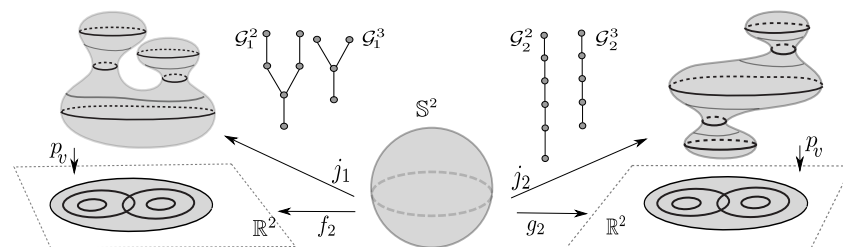

Figure 2: Examples of $\mathcal{M}$-graph and $\mathcal{F}$-graph type tree.

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Definition 3.2. *Let $f = (f_2, f_3) : M \to \mathbb{R}^2 \times \mathbb{S}^2$ be a stable plane-Gauss map. We denote the graph of f by $(\mathcal{G}^2, \mathcal{G}^3)$, where $\mathcal{G}^2$ is the graph of f_2 and $\mathcal{G}^3$ the graph of f_3.*

Definition 3.3. *We say that the pair of graphs $(\mathcal{G}, \mathcal{H})$ is a $\mathcal{MF}$-graph if exists a stable plane-Gauss map $f = (f_2, f_3) : M \to \mathbb{R}^2 \times \mathbb{S}^2$ such that the graph of f_2 is $\mathcal{G}$ the graph of f_3 is $\mathcal{H}$.*

Given a closed and oriented surface M, by Definition 3.2, every stable plane-Gauss map $(f_2, f_3) : M \to \mathbb{R}^2 \times \mathbb{S}^2$ is associated with an $\mathcal{MF}$-graph. Some natural questions can be considered:

  i) Is every graph $\mathcal{GH}$ an $\mathcal{MF}$-graph?

  ii) Otherwise, what conditions should we impose on a graph $\mathcal{GH}$ in order to be an $\mathcal{MF}$-graph?

The motivation of the study on these pairs of graphs is find answers to these questions, among others. We will see some constructions of these maps that can lead to future answers to these questions in the general case.

4 Lips and beaks transitions of plane-Gauss maps

We can always obtain a new stable plane-Gauss map, associated with new $\mathcal{MF}$-graphs by changing the immersion of M in $\mathbb{R}^3$. These alterations can be made changing the Gauss maps $f_3 : M \to \mathbb{S}^2$ and can change at the same time, or not, the image of the $f_2 : M \to \mathbb{R}^2$, depending on the map in the plane or changing f_2 without varying f_3.

A *codimension one transition* corresponds to the intersection of the set Δ, of non-stable maps, with generic isotopy from a given stable map f_i to another stable map g_i ($i = 2, 3$) that are in different path-components of $\mathcal{E}(M, N)$. In other words, this means that codimension one transition is the point at which a path between f_i and g_i transversely intersects a strata of Δ. The types of transitions are described in [8], in the case of surfaces in the plane and the case of Gauss maps ([3,7]). For lips and beaks transitions that changes the number of cusps of each f_i, the edge number of the graph, the number of vertices or the weight of the graphs, fulfills $g(M) = 1 - V^i + E^i + W^i$ ($i = 2, 3$) is constant. For more details, see [6,7].

Definition 4.1. *A cusp point $x \in \Sigma f_i$ ($i = 2, 3$) is called positive (resp. negative) if its local mapping degree, in a neighborhood U_x of x is $+1$ (resp. -1) with respect to the given orientations.*

i) **Lips transition $\mathbf{L}_i^{\pm}$ of map** f_i ($i = 2, 3$): increases the number of cusps and increases by 1 the number of singular curves of f_i: $\mathbf{L}_i^+$ increases by 1 the number of regions negative and $\mathbf{L}_i^-$ increases by 1 the number of regions positive. Consequently, $\mathbf{L}_i^{\pm}$ increases one vertex and one edge on the graph $\mathcal{G}^i$.

ii) **Beaks transition $\mathbf{B}_i$ of map** f_i ($i = 2, 3$): increases the number of cusps. Beaks transitions of f_i can be classified in eight different transitions that change the graph $\mathcal{G}^i$ (see Figure 3):

    $B_{vi}^{+,\pm}$: increases by 1 the number of vertices in $V^{i\pm}$ and the number of edges E^i.

    $B_{vi}^{-,\pm}$: decreases by 1 the number of vertices in $V^{i\pm}$ and the number of edges E^i.

    $B_{wi}^{+,\pm}$: increases by 1 the weight in $W^{i\pm}$ and decreases by 1 the number of edges E^i.

    $B_{wi}^{-,\pm}$: decreases by 1 the weight in $W^{i\pm}$ and increases by 1 the number of edges E^i.

A transition $\delta \in \{L_i^{\pm}, B_{vi}^{\pm,\pm}, B_{wi}^{\pm,\pm}\}$ that decreases the number of cusps, will be denoted by $-\delta$.

Figure 3 illustrates a sequence of transitions on the sphere that simultaneously change the $\mathcal{M}$-graph and the $\mathcal{F}$-graph.

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Remark 4.1. *The beaks and lips transitions for maps in the plane, can change or not the $\mathcal{F}$-graph (see Figure 3). Similarly, beaks and lips for Gauss maps can change or not the $\mathcal{M}$-graph, as illustrates the Figure 4.*
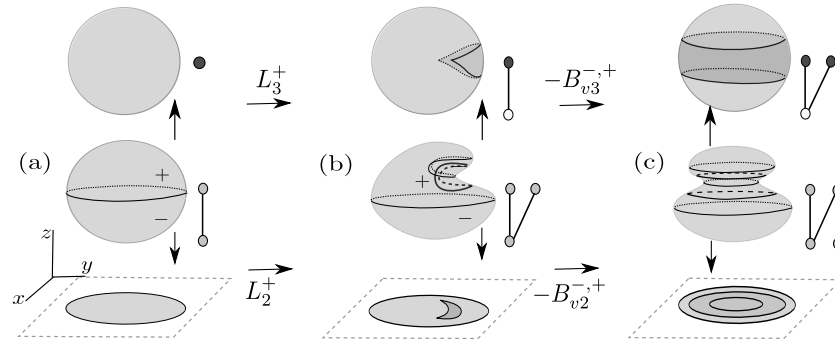

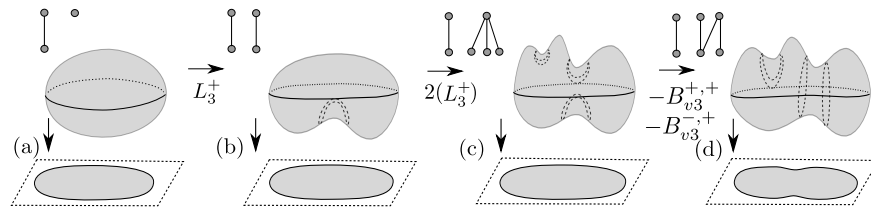Figure 3: Transitions that change the $\mathcal{M}$-graph and $\mathcal{F}$-graph.

Figure 4: Transitions that only change the $\mathcal{F}$-graph.

Remark 4.2. *Note that the sequence of transitions, in Figure 4, does not change the $\mathcal{M}$-graph it also does not change the apparent contour. This shows that the graph of the Gaussian map is an invariant that helps to refine the classification of maps in the plane.*
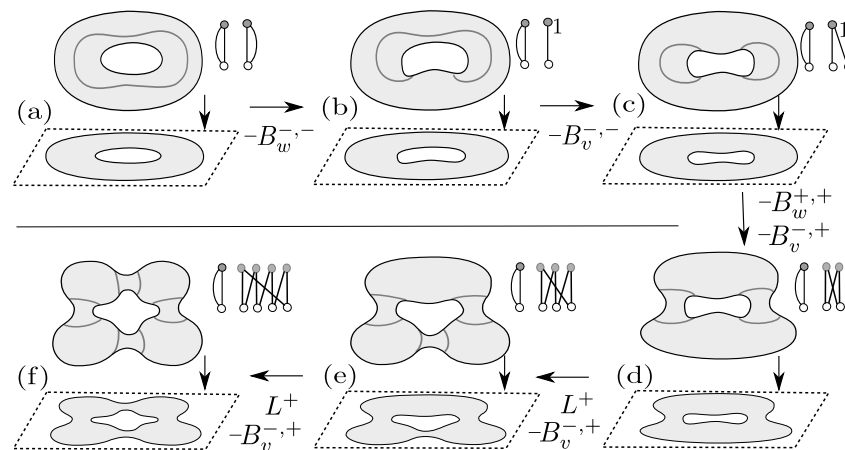

Figure 5: Example of $\mathcal{MF}$-graphs with cycles.

Figure 5 shows a sequence of stable maps of the torus with their graphs with transitions that changes the topology of parabolic curves keeping invariant the topology of map curves in the plane. In: (a) $(\mathcal{G}^2(2,2,0), \mathcal{G}^3(2,2,0))$, each parabolic curve of f_3 has 4 cusp points: two positive

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and two negative (see [7]); (b) is obtained from (a) by transitions $-B_{w3}^{-;-}$; it removes the pairs of positive cusps of f_3, leaving the new map with a elliptic region of genus one and one hyperbolic regions simply connected; (c) is obtained from (b) by transitions $-B_{v3}^{-;-}$; dividing the parabolic curve and the hyperbolic region into two components; (d) is obtained from (c) by transitions $-B_{w3}^{+;+}$, which decomposes a parabolic curve and removes the genus from the elliptic region and is followed by $-B_{v3}^{-;+}$ which decomposes the following parabolic curve, obtaining a map with 4 parabolic curves that separates two elliptic regions from two hyperbolic regions, all homeomorphic to the cylinder and with zero weight graph $(\mathcal{G}^2(2,2,0), \mathcal{G}^3(4,4,0))$; (e) is obtained from (d) by transitions L_3^+ followed by $-B_{v3}^{-;+}$, creating a new elliptic and hyperbolic region with the graph $(\mathcal{G}^2(2,2,0), \mathcal{G}^3(6,6,0))$; the same happens with the transitions from (e) to (f) that realizes the graph $(\mathcal{G}^2(2,2,0), \mathcal{G}^3(8,8,0))$. We can obtain a stable map that realizes the graph $(\mathcal{G}^2(2,2,0), \mathcal{G}^3(V,V,0))$, without changing the $\mathcal{M}$-graph in the following way (see Figure 5-(e) and (f)).

5 Realization of some trees with total weight zero

In this section we present some families of graphs $\mathcal{G}^2\mathcal{G}^3$ that are $\mathcal{MF}$-graphs, that is, graphs that can be realized by stable plane-Gauss maps in the sense that there is a smooth connected closed orientable surface M and a stable plane-Gauss map $f = (f_2, f_3) : M \to \mathbb{R}^2 \times \mathbb{S}^2$ such that the graph of f is $(\mathcal{G}^2, \mathcal{G}^3)$. This will be done through beaks transitions and lips on smaller maps already known. The results here will only be valid for zero weight graphs. For weight greater than zero will be treated in later works.

Before introducing the general result tree-type graphs (which satisfies $V = E + 1$) with zero weight and some graphs with cycles in general, we will see some examples of realization. $(\mathcal{G}^2(2,1,0), \mathcal{G}^3(1,0,0))$ can be realized (see Figure 4-(a)) by a $\mathbb{S}^2$ embedding in the 3-space with all elliptic points, which projected in a plane has a unique singular curve, which separates two regular regions. $(\mathcal{G}^2(2,1,0), \mathcal{G}^3(2,1,0))$ can be realized as Figure 4-(b), by lip transition L_3^+ on (a) on one of the regular regions f_2, without changing Σf_2.

Definition 5.1. *The degree of a vertex v in a graph is the number of edges incident to it. A tree it is called star positive (negative) if it has $V - 1$ negative vertices (positive) with degree 1.*

Theorem 5.1. *The pair of graphs $(\mathcal{G}^2(2,1,0), \mathcal{G}^3(V, V-1, 0))$ is an $\mathcal{MF}$-graph, for all $V > 0$.*

Proof. We have shown that $(\mathcal{G}^2(2,1,0), \mathcal{G}^3(1,0,0))$ is an $\mathcal{MF}$-graph. To realize the couple $\mathcal{G}^2(2,1,0)\mathcal{G}^3(V, V-1, 0)$ remove a negative (or positive) vertex, of degree 1, from $\mathcal{G}^3(V, V-1, 0)$ and suppose the pair $(\mathcal{G}(2,1,0), \mathcal{G}(V-1, V-2, 0))$ realizes by a map $(g_2, g_3) : \mathbb{S}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{S}^2$. Through the lips transition L_3^+ (or L_3^-) on (g_2, g_3), without changing the Σf_2 (Remark 4.1), we can get a map (f_2, f_3) which realizes the pair with $\mathcal{G}^3(V, V-1, 0)$. $\square$

Corollary 5.1. *The pair of graphs $(\mathcal{G}^2(2,1,0), \mathcal{G}^3(V, V-1, 0))$, where $\mathcal{G}^3$ is a star with $V > 0$, is an $\mathcal{MF}$-graph.*

Theorem 5.2. *Every pair of graphs $(\mathcal{G}^2(V, V-1, 0), \mathcal{G}^3(2,1,0))$, with $V > 1$, is an $\mathcal{MF}$-graph.*

Proof. We have shown that $(\mathcal{G}^2(2,1,0), \mathcal{G}^3(2,1,0))$ is an $\mathcal{MF}$-graph. To realize the couple $(\mathcal{G}^2(V, V-1, 0), \mathcal{G}^3(2,1,0))$ remove a negative (or positive) vertex, of degree 1, from $\mathcal{G}^2(V, V-1, 0)$ and suppose the pair $(\mathcal{G}^2(V-1, V-2, 0), \mathcal{G}^3(2,1,0))$ realizes by a map $(g_2, g_3) : \mathbb{S}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{S}^2$. Through the lips transition L_2^+ (or L_2^-) on (g_2, g_3), without changing the Σf_3, we can get a map (f_2, f_3) which realizes the pair with $\mathcal{G}^2(V, V-1, 0)$. $\square$

Corollary 5.2. *The pair of graphs $(\mathcal{G}^2(V, V-1, 0), \mathcal{G}^3(2,1,0))$, where $\mathcal{G}^2$ is a star with $V > 1$, is an $\mathcal{MF}$-graph.*

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Theorem 5.3. *All graph* $(\mathcal{G}^2(V^2, V^2 - 1, 0), \mathcal{G}^3(V^3, V^3 - 1, 0))$, *with* $V^2, V^3 > 1$, *is an* $\mathcal{MF}$-*graph.*

Proof. $(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(V^3, V^3 - 1, 0))$ can be realized by a map (f_2^1, f_3^1) as Theorem 5.1. Then, chooses the hyperbolic and elliptical regions corresponding to vertices the u_i and v_i in the neighborhood of the parabolic curve associated with edges $u_i v_i$, of $\mathcal{G}^3(V^3, V^3 - 1, 0)$. Lips transitions can be applied to one of the regular regions of (f_2^1, f_3^1), as in Theorem 5.2, properly, to obtain a map (f_2, f_3) that realizes the graph $(\mathcal{G}^2(V^2, V^2 - 1, 0), \mathcal{G}^3(V^3, V^3 - 1, 0))$, without changing the topology of the singular set of f_3. $\qquad\square$

6   Conclusions

All pair of graphs $(\mathcal{G}^2(V^2, V^2 - 1, 0), \mathcal{G}^3(V^3, V^3 - 1, 0))$, with $V^2, V^3 > 1$, can be associated to an immersion $j : \mathbb{S}^2 \to \mathbb{R}^3$, where are stable the projection $p_v : j(\mathbb{S}^2) \to \mathcal{P} \subset \mathbb{R}^3$ ($\mathcal{P}$ is a orthogonal plane to $v \in \mathbb{R}^3$ and p_v is the restriction to $j(\mathbb{S}^2)$, of the projection of $\mathbb{R}^3$ in $\mathcal{P}$) and the Gauss map $\mathcal{N}_j : j(\mathbb{S}^2) \to \mathbb{S}^2$ of $j(\mathbb{S}^2)$.

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