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Fractional differential equations: Ulam-Hyers stabilities

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Abstract. Since the first work on Ulam-Hyers stabilities of differential equation solutions to date, many important and relevant papers have been published, both in the sense of integer order and fractional order differential equations. However, when we enter the field of fractional calculus, in particular, involving fractional differential equations, the path that is still long to be traveled, although there is a range of published works. In this sense, in this paper, we will investigate the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of mild solutions of the fractional nonlinear abstract Cauchy problem in the intervals $[0,T]$ and $[0,\infty)$, by means of Banach's fixed point theorem.

Key-words. Fractional nonlinear abstract Cauchy, Ulam-Hyers stabilities, mild solution, Banach fixed point theorem..

1 Introduction

In 2012, Wang and Zhou [10] in their work, investigated several kind of stabilities of the mild solution stability of the fractional evolution equation in Banach space, namely: Mittag-Leffler-Ulam stability, Mittag-Leffler-Ulam-Hyers stability, Mittag-Leffler-Ulam-Hyers-Rassias stability and generalized Mittag-Leffleer-Ulam-Hyers-Rassias stability. Zhou and Jiao [11], using fractional operators and some fixed point theorems, investigated the existence and uniqueness of mild solutions of fractional neutral evolution equations and made some applications in order to elucidate the obtained results. In this sense, Saadati et. al. [2], presented results on the existence of mild solutions for fractional abstract equations with non-instantaneous impulses. In order to obtain such results, the authors used non-compactness measure and the Darbo-Sadovskii and Tichonov fixed point theorems.

Although we are faced with a significant amount of work dealing with solution properties of fractional differential equations, there is still much work to be done. In order to propose new results and provide new materials on Ulam-Hyers stability in order to contribute positively to the area, the present work has as main objective to investigate some Ulam-Hyers stabilities in the intervals $[0,T]$ and $[0,\infty)$.

So let's consider the fractional nonlinear abstract Cauchy problem given by

$$\left\{\begin{array}{rcl} {}^{H}\mathbb{D}^{\alpha,\beta}_{0^+}\xi(t) & = & \mathcal{A}\xi(t)+u(t)\mathcal{H}(t,\xi(t)), \qquad t\in I \\ I^{1-\gamma}_{0^+}\xi(0) & = & \xi_0 \end{array}\right. \tag{1}$$

where ${}^{H}\mathbb{D}^{\alpha,\beta}_{0^+}(\cdot)$ is the Hilfer fractional derivative of order $0<\alpha\leq 1$ and type $0\leq\beta\leq 1$, $I=[0,T]$ or $[0,\infty)$, $\xi\in\Omega$, Ω Banach space, $t\in I$, $:\Omega\to\Omega$ is the infinitesimal generator of a C_0-semigroup $(\mathbb{S}(t))_{t\geq 0}$ and $\mathcal{H}: I\times\Omega\to\Omega$ is a given continuous function.

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The work is organized as follows. In section 2 we introduce the ψ-Riemann-Liouville fractional integral, the ψ-Hilfer fractional derivative and fundamental concept of the operator (α,β)-resolvent. In this sense, it is presented the mild solution of the fractional Cauchy problem as well as the Ulam-Hyers stability. In section 3, it is directed to the first result of this work, that is, we investigate the Ulam-Hyers and Ulam-Hyers-Rassias stabilities in the $[0,T]$ range and discuss some particular cases.

2 Preliminaries

Let $T>0$ be a given positive real number. The weighted space of continuous functions $\xi\in I_1=(0,T]$ and $I=[0,T]$ is given by [4]

$$C_{1-\gamma}(I_1,\Omega)=\left\{\xi\in C(I_1,\Omega),\, t^{1-\gamma}\xi(t)\in C(I_1,\Omega)\right\}$$

where $0\leq\gamma\leq 1$, with norm

$$||\xi||_{C_{1-\gamma}} = \sup_{t\in I}||\xi(t)||_{C_{1-\gamma}}$$

and

$$||\xi-\phi||_{C_{1-\gamma}} = \mathrm{d}_{1-\gamma}(\xi,\phi):=\sup_{t\in I}||\xi(t)-\phi(t)||_{C_{1-\gamma}}.$$

Let $(\Omega,||\cdot||_{C_{1-\gamma}})$ be a given Banach space and $I=[0,+\infty)$ or $I=[0,T]$ where T and $\mathscr{L}(\Omega)$ the set of bounded linear maps from Ω to Ω.

Let (a,b) $(-\infty\leq a<b\leq\infty)$ be a finite interval (or infinite) of the real line $\mathbb{R}$ and let $\alpha>0$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ (we denote first derivative as $\dfrac{d}{dx}\psi(x)=\psi'(x)$) on (a,b). The left-sided fractional integral of a function f with respect to a function ψ on $[a,b]$ is defined by [4,5]

$$\mathcal{I}_{a+}^{\alpha;\psi}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(s)\left(\psi(x)-\psi(s)\right)^{\alpha-1}f(s)\,ds. \tag{2}$$

On the other hand, let $n-1<\alpha<n$ with $n\in\mathbb{N}$, let $J=[a,b]$ be an interval such that $-\infty\leq a<b\leq\infty$ and let $f,\psi\in C^n[a,b]$ be two functions such that ψ is increasing and $\psi'(x)\neq 0$, for all $x\in J$. The left-sided $\psi-$Hilfer fractional derivative ${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ of a function f of order α and type $0\leq\beta\leq 1$, is defined by [5]

$${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x)=\mathcal{I}_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathcal{I}_{a+}^{(1-\beta)(n-\alpha);\psi}f(x). \tag{3}$$

Next, we present the definition of the fundamental operator (α,β)-resolvent in the presentation of the mild solution of the fractional abstract Cauchy problem Eq.(1).

Definition 2.1. [3] *Let $\alpha>0$ and $\beta\geq 0$. A function $\mathbb{S}_{\alpha,\beta}:\mathbb{R}_+\to\mathscr{L}(\Omega)$ is called a β-times integrated α-resolvent operator function of an (α,β)-resolvent operator function* (ROF) *if the following conditions are satisfied:*

1. *$\mathbb{S}_{\alpha,\beta}(\cdot)$ is strongly continuous on $\mathbb{R}_+$ and $\mathbb{S}_{\alpha,\beta}(0)=g_{\beta+1}(0)I$;*

2. *$\mathbb{S}_{\alpha,\beta}(s)\mathbb{S}_{\alpha,\beta}(t)=\mathbb{S}_{\alpha,\beta}(t)\mathbb{S}_{\alpha,\beta}(s)$ for all $t,s\geq 0$;*

3. *the function equation*
$\mathbb{S}_{\alpha,\beta}(s)I_t^\alpha\mathbb{S}_{\alpha,\beta}(t) - I_s^\alpha\mathbb{S}_{\alpha,\beta}(s)\mathbb{S}_{\alpha,\beta}(t) = g_{\beta+1}(s)I_t^\alpha\mathbb{S}_{\alpha,\beta}(t) - g_{\beta+1}(t)I_s^\alpha\mathbb{S}_{\alpha,\beta}(s)$ *for all* $t,s\geq 0$.

The generator $\mathcal{A}$ of $\mathbb{S}_{\alpha,\beta}$ is defined by

$$D(\mathcal{A}) := \left\{x\in\Omega : \lim_{t\to 0^+}\frac{\mathbb{S}_{\alpha,\beta}(t)\,x - g_{\beta+1}(t)\,x}{g_{\alpha+\beta+1}(t)} \text{ exists}\right\} \tag{4}$$

and

$$\mathcal{A}\,x := \lim_{t\to 0^+}\frac{\mathbb{S}_{\alpha,\beta}(t)\,x - g_{\beta+1}(t)\,x}{g_{\alpha+\beta+1}(t)}\,, \quad x\in D(\mathcal{A}), \tag{5}$$

where $g_{\alpha+\beta+1}(t) := \dfrac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta)}$ $(\alpha+\beta>0)$.

Now, we consider the continuous function given $\mathcal{H}: I\times\Omega\to\Omega$ such that, for almost all $t\in I$, we get

$$||\mathcal{H}(t,x)-\mathcal{H}(t,y)||_{C_{1-\gamma}} \leq \ell(t)||x-y||_{C_{1-\gamma}}\,, \quad x,y\in\Omega \tag{6}$$

where $\ell:[0,T]\to\mathbb{R}^+$ and $u:[0,T]\to\mathbb{R}$ are two given measurable functions such that ℓ, u and ℓu are locally integrable on I.

The following is the definition of the Mainardi function, fundamental in mild solution of the Eq.(1). Then, the Wright function, denoted by $M_\alpha(Q)$, is defined by [1,7]

$$M_\alpha(Q) = \sum_{n=1}^{\infty}\frac{(-Q)^{n-1}}{(n-1)!\Gamma(1-\alpha n)}\,, \quad 0<\alpha<1, \quad Q\in\mathbb{C}$$

satisfying the relation

$$\int_0^\infty \theta^{\overline{\delta}}M_\alpha(\theta)\,\mathrm{d}\theta = \frac{\Gamma(1+\overline{\delta})}{\Gamma(1+\alpha\overline{\delta})}\,, \quad \text{for } \theta,\overline{\delta}\geq 0\cdot$$

Lemma 2.1. [1,7] *The fractional nonlinear differential equation,* Eq.(1)*, is equivalent to the integral equation*

$$\xi(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}\xi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\mathcal{A}\xi(s) + u(s)\,\mathcal{H}(s,\xi(s))\right]\,\mathrm{d}s\,,\ t\in[0,T]\cdot \tag{7}$$

A function $\xi\in C_{1-\gamma}(I,\Omega)$ is called a mild solution of Eq.(1)*, if the integral equation,* Eq.(7) *holds, we have*

$$\xi(t) = \mathbb{S}_{\alpha,\beta}(t)\xi(0) + \int_0^t \mathbb{T}_\alpha(t-s)u(s)\mathcal{H}(s,\xi(s))\,\mathrm{d}s\,, \quad t\in I \tag{8}$$

where $\mathbb{T}_\alpha(t) = t^{\alpha-1}G_\alpha(t)$, $G_\alpha(t) = \displaystyle\int_0^\infty \alpha\theta M_\alpha(\theta)\mathbb{S}(t^{\alpha\theta})\,\mathrm{d}\theta$ *and* $\mathbb{S}_{\alpha,\beta}(t) = \mathcal{I}_0^{\beta(1-\alpha)}\mathbb{T}_\beta(t)$.

For a given $\xi_0\in\Omega$ and any $\xi\in C_{1-\gamma}(I,\Omega)$, we set

$$\Lambda(\xi)(t) := \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t \mathbb{T}_\alpha(t-s)u(s)\mathcal{H}(s,\xi(s))\,\mathrm{d}s \tag{9}$$

for all $t\in I$.

For the procedure in this paper, ℓ, u are measurable functions such that ℓ, u and the product ℓu are locally integrable. Moreover, it is easy to see that the application $\xi\to\Lambda(\xi)$ is a self-mapping of the space $C_{1-\gamma}(I,\Omega)$.

On the other hand, for $\xi_0 \in \Omega$ and ε, we consider

$$\xi(t) = \Lambda(\xi(t)), \quad t \in I \tag{10}$$

and the following inequalities

$$||\xi(t) - \Lambda(\xi(t))||_{C_{1-\gamma}} \leq \varepsilon\,, \quad t \in I \tag{11}$$

and

$$||\xi(t) - \Lambda(\xi(t))||_{C_{1-\gamma}} \leq G(t)\,, \quad t \in I \tag{12}$$

where $\xi \in C_{1-\gamma}(I,\Omega)$ and $G \in C_{1-\gamma}(I,(0,+\infty))$.

The following are the definitions of the main results to be investigated in this paper. The definitions were adapted to the problem version of fractional differential equations. Then we have:

Definition 2.2. [3, 10] *The* Eq.(10) *is Ulam-Hyers stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $\xi \in C_{1-\gamma}(I,\Omega)$ of* Eq.(10) *such that*

$$||\xi(t) - v(t)||_{C_{1-\gamma}} \leq \varepsilon\,, \quad t \in I\cdot \tag{13}$$

Definition 2.3. [3, 10] *The* Eq.(10) *is generalized Ulam-Hyers stable if there exists $\theta \in C_{1-\gamma}([0,+\infty),[0,+\infty))$, $\theta(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $\xi \in C_{1-\gamma}(I,\Omega)$ of* Eq.(11) *there exists a solutions $v \in C_{1-\gamma}(I,\Omega)$ of* Eq.(10) *such that*

$$||\xi(t) - v(t)||_{C_{1-\gamma}} \leq \theta(\varepsilon)\,, \quad t \in I\cdot \tag{14}$$

Definition 2.4. [3, 10] *The* Eq.(10) *is generalized Ulam-Hyers-Rassias stable with respect to $G \in C_{1-\gamma}([0,+\infty),[0,+\infty))$, if there exists $c_G > 0$ such that for each solution $\xi \in C_{1-\gamma}(I,\Omega)$ of* Eq.(12) *there exists a solution $v \in C_{1-\gamma}(I,\Omega)$ of* Eq.(10) *such that*

$$||\xi(t) - v(t)||_{C_{1-\gamma}} \leq c_G G(t)\,, \quad t \in I\cdot \tag{15}$$

3 Ulam-Hyers and Ulam-Hyers-Rassias stabilities of mild on $[0,T]$.

Let $(\mathbb{S}_{\alpha,\beta}(t))_{t\geq 0}$ the (α,β)-resolvent operator function on a Banach space $(\Omega, ||\cdot||_{C_{1-\gamma}})$. and the continuous function $\xi : [0,T] \to \Omega$, given by

$$\Lambda(\xi)(t) := \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t \mathbb{T}_\alpha(t-s)u(s)\mathcal{H}(s,\xi(s))\,\mathrm{d}s\,, \quad t \in [0,T) \tag{16}$$

for $\xi_0 \in \Omega$ fixed.

Then, we have the theorem that gives certain conditions, guarantees the Ulam-Hyers stability to Eq.(10) on the finite interval $[0,T]$.

Theorem 3.1. *Let $(\mathbb{S}_{\alpha,\beta}(t))_{t\geq 0}$ the (α,β)-resolvent operator function on a Banach space $(\Omega, ||\cdot||_{C_{1-\gamma}})$, with $0 \leq \gamma \leq 1$ and let $T > 0$ be a positive real number. We set*

$$\widetilde{\lambda} := \delta \int_0^T e^{w(T-s)}|u(s)|\ell(s)\,\mathrm{d}s\cdot \tag{17}$$

If $\widetilde{\lambda} < 1$, then the Eq.(10) *is stable in the Ulam-Hyers sense.*

Thus, we conclude the first part of the result. Next, we will investigate the Ulam-Hyers-Rassias stability by completing the first purpose of this paper.

Theorem 3.2. *Let $(\Omega, ||\cdot||_{C_{1-\gamma}})$ be a Banach space and let $(\mathbb{S}_{\alpha,\beta}(t))_{t\geq 0}$ be a (α,β)-resolvent operator function on Ω. Let $\delta \geq 1$, $w \geq 0$ be constants such that*

$$||\mathbb{S}_{\alpha,\beta}(t)||_{C_{1-\gamma}} \leq \delta\, e^{wt} \text{ and } ||\mathbb{T}_{\alpha}(t)||_{C_{1-\gamma}} \leq \delta\, e^{wt} \tag{18}$$

for all $t \geq 0$. Let $\xi_0 \in \Omega$, $T > 0$ and $G : [0,T] \to (0,\infty)$ be a continuous function.

Suppose that a continuous function $f : [0,T] \to \Omega$ satisfies

$$\left|\left| f(t) - \mathbb{S}_{\alpha,\beta}(t)\xi_0 - \int_0^t \mathbb{T}_{\alpha}(t-s)u(s)\mathcal{H}(s,f(s))\,\mathrm{d}s \right|\right|_{C_{1-\gamma}} \leq G(t) \tag{19}$$

for all $t \in [0,T]$.

Suppose that there exists a positive constant ρ such that

$$\ell(s)|u(s)|e^{w(T-s)} \leq \rho \tag{20}$$

for almost all $s \in [0,T]$. Then, $\exists C_G > 0$ (constant) and a unique continuous function $v : [0,T] \to\to \Omega$ such that

$$v(t) = \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t \mathbb{T}_{\alpha}(t-s)u(s)\mathcal{H}(s,v(s))\,\mathrm{d}s\,, \quad t \in [0,T] \tag{21}$$

and

$$||f(t) - v(t)||_{C_{1-\gamma}} \leq C_G G(t)\,, \quad t \in [0,T]\cdot \tag{22}$$

Demonstração. Consider $K > 0$ be such that $K\delta\rho < 1$ and continuous function $\phi : [0,T] \to (0,\infty)$ as follows,

$$\int_0^t \phi(s)\,\mathrm{d}s \leq K\,\phi(t)\,, \quad t \in [0,T]. \tag{23}$$

Now let, f, G satisfy the inequality (19) and $\widetilde{\alpha}_G, \widetilde{\beta}_G > 0$ such that

$$\widetilde{\alpha}_G\phi(t) \leq G(t) \leq \widetilde{\beta}_G\phi(t)\,, \quad t \in [0,T]\cdot \tag{24}$$

On the other hand, for all $h, g \in C_{1-\gamma}(I,\Omega)$, consider the following set

$$\mathrm{d}_{\phi,1-\gamma}(h,g) := \inf\left\{C \in [0,\infty) : ||h(t) - g(t)||_{C_{1-\gamma}} \leq C\phi(t)\,, \quad t \in [0,T]\right\}\cdot$$

It is easy to see that $(C_{1-\gamma}(I,\Omega), \mathrm{d}_{\phi,1-\gamma})$ is a metric and that $(C_{1-\gamma}(I,\Omega), \mathrm{d}_{\phi,1-\gamma})$ is a complete metric space.

Now, consider the operator $\Lambda : C_{1-\gamma}(I,\Omega) \to C_{1-\gamma}(I,\Omega)$ defined by

$$(\Lambda h)(t) := \mathbb{S}_{\alpha,\beta}(t)\xi_0 + \int_0^t \mathbb{T}_{\alpha}(t-s)u(s)\mathcal{H}(s,h(s))\,\mathrm{d}s\,, \quad t \in [0,T]\cdot$$

The next step is to show that Λ is a contraction in the metric space $C_{1-\gamma}(I,\Omega)$ induced by metric $\mathrm{d}_{\phi,1-\gamma}$. Then, let $h, g \in C_{1-\gamma}(I,\Omega)$ and $C(h,g) \in [0,\infty)$ a constant such that

$$||h(t) - g(t)||_{C_{1-\gamma}} \leq C(h,g)\phi(t)\,, \quad t \in [0,T]\cdot$$

Then, using Eq.(18), Eq.(20) and Eq.(23), we obtain

$$||(\Lambda h)(t) - (\Lambda g)(t)||_{C_{1-\gamma}} \leq C(h,g)\delta\rho K\phi(t)\,, \quad t \in [0,T]\cdot$$

Therefore, we have $d_{\phi,1-\gamma}(\Lambda(h),\Lambda(g)) \leq \delta\rho K C(h,g)$ from which we deduce that

$$d_{\phi,1-\gamma}(\Lambda(h),\Lambda(g)) \leq \delta\rho K d_{\phi,1-\gamma}(h,g)\cdot$$

Using the fact that $\delta\rho K < 1$, we have that Λ is a contraction in $(C_{1-\gamma}(I,\Omega), d_{\phi,1-\gamma})$. In this sense, through Banach's fixed point theorem, we have that there is a unique function $v \in C_{1-\gamma}(I,\Omega)$ such that $v = \Lambda(v)$. Now, using By the triangle inequality, we get

$$\begin{array}{rcl} d_{\phi,1-\gamma}(f,v) & \leq & d_{\phi,1-\gamma}(f,\Lambda(f)) + d_{\phi,1-\gamma}(\Lambda(f),\Lambda(v)) \\ & \leq & \beta_G + \delta\rho K d_{\phi,1-\gamma}(f,v) \end{array}$$

which implies that

$$d_{\phi,1-\gamma}(f,v) \leq \frac{\beta_G}{1-\delta\rho K}. \tag{25}$$

Which in turn, we obtain

$$\begin{array}{rcl} ||f(t)-v(t)||_{C_{1-\gamma}} & \leq & \dfrac{\beta_G}{1-\delta\rho K}\phi(t) \\ & \leq & \dfrac{\beta_G}{1-\delta\rho K}\dfrac{G(t)}{\alpha_G} = C_G G(t)\,, \quad t \in [0,T] \end{array} \tag{26}$$

where $C_G := \dfrac{\beta_G}{(1-\delta\rho K)\alpha_G}$, which is the desired inequality (22). $\square$

4 Concluding remarks

We conclude this work with the objectives achieved, that is, we investigate the Ulam-Hyers and Ulam-Hyers-Rassias stabilities for the mild solution of the fractional nonlinear abstract non-linear Cauchy problem: the first part was destined to the limited interval $[0,T]$ and the second part to the interval $[0,\infty)$. It is important to emphasize the fundamental role of the Banach fixed point theorem in the results obtained.

Although, the results presented here, contribute to the growth of the theory; some questions still need to be answered. The first question is about the possibility of investigating the existence and uniqueness of mild solutions for fractional differential equations formulated via ψ-Hilfer fractional derivative.

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Referências

[1] Gu, H. and Trujillo, J. J., Existence of mild solution for evolution equation with Hilfer fractional derivative, Appl. Math. Comput., 2015, 257:344–354.

[2] Saadati, R., Pourhadi, E., Samet, B., On the $\mathcal{PC}$-mild solutions of abstract fractional evolution equations with non-instantaneous impulses via the measure of noncompactness, Bound. Value Probl., 2019, 2019:1.

[3] Sousa, J. Vanterler da C., Kucche, K. D., Oliveira, E. Capelas de, Ulam-Hyers stabilities of mild solutions of the fractional nonlinear abstract Cauchy problem, Nonauton. Dyn. Sys., (Submitted) (2019).

[4] Sousa, J. Vanterler da C., Kucche, K. D., Oliveira, E. Capelas de, Stability of ψ-Hilfer impulsive fractional differential equations, Appl. Math. Lett., 2019, 88:73–80.

[5] Sousa, J. Vanterler da C., Oliveira, E. Capelas de , On the ψ-Hilfer fractional derivative, Commun. Nonlinear Sci. Numer. Simulat., 2018, 60:72–91.

[6] Sousa, J. Vanterler da C., Oliveira, E. Capelas de , Leibniz type rule: ψ-Hilfer fractional operator, Commun. Nonlinear Sci. Numer. Simul., 2019, 77:305–311.

[7] Sousa, J. Vanterler da C., Existence of mild solutions to Hilfer fractional evolution equations in Banach space, arXiv:1812.02213, (2018).

[8] Sousa, J. Vanterler da C., Oliveira, D. S., Oliveira, E. Capelas de, A note on the mild solutions of Hilfer impulsive fractional differential equations, Chaos, Solitons, & Fractals, 2021, 147, 110944.

[9] Ulam, S. M., A collection of mathematical problems, 8 (1960).

[10] Wang, J., Zhou, Y., Mittag-Leffler-Ulam stabilities of fractional evolution equations, Appl. Math. Lett., 2012, 25(4):723–728

[11] Zhou, Y., Jiao, F., Existence of mild solutions for fractional neutral evolution equations, Comput. Math. Appl., 2010, 59(3): 1063–1077.

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