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On attractivity of solutions of fractional differential equations

J. Vanterler da C. Sousa [1]

Department of Applied Mathematics, Imecc-Unicamp, 13083-859, Campinas, SP, Brazil

E. Capelas de Oliveira [2]

Department of Applied Mathematics, Imecc-Unicamp, 13083-859, Campinas, SP, Brazil

Abstract. In this work, we investigate the existence of a class of globally attractive solutions of the Cauchy fractional problem with the ψ-Hilfer fractional derivative using the measure of noncompactness.

Key-words. Fractional differential equations, ψ-Hilfer fractional derivative, attractivity, measure noncompactness.

1 Introduction

Consider the following Cauchy fractional problem

$$\begin{cases} {}^{H}\mathscr{D}_{0+}^{\nu,\eta;\psi}\theta(t) &= u(t,\theta(t)), \quad t\in(0,\infty) \\ I_{0+}^{1-\gamma;\psi}\theta(0) &= \theta_0 \end{cases} \tag{1}$$

where ${}^{H}\mathscr{D}_{0+}^{\nu,\eta;\psi}\theta(\cdot)$ is the ψ-Hilfer fractional derivative of order $0<\nu<1$ and type $0\leq\eta\leq 1$, $\mathcal{I}_{0+}^{1-\gamma;\psi}\theta(\cdot)$ is the fractional integral of order γ, with $0\leq\gamma<1$ with respect to another function, $u:[0,\infty)\times\Omega\to\Omega$ is a continuous function satisfying some conditions and θ_0 is a element of the Banach space.

The theory of fractional differential equations can be found in for example [3,7,11]. Existence, uniqueness and Ulam-Hyers stabilities of solutions of differential and integrodifferential equations was studied using the ψ-Hilfer fractional derivative in [6,7]. In 2018, Sousa and Oliveira [6], investigated the Ulam-Hyers stability of an fractional integrodifferential equation using the Banach fixed point theorem. We also refer the reader to [3, 7, 10, 11]. Attractivity of mild solutions of fractional differential and integrodifferential equations was considered in [10, 11]. Chang et al. [2], investigated the asymptotic decay of some operators via fixed point theorems and they considered the existence and uniqueness for a class of mild solutions of Sobolev fractional differential equations. In 2008 Banas and O'Regan [1] investigated the existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order in Banach spaces and in 2012 Chen et al. [3]

[1]vanterler@ime.unicamp.br.

[2]capelas@unicamp.br.

considered the global attractivity of solutions of fractional differential equations in the Riemann-Liouville fractional derivative sense, using the Krasnoselskii fixed-point theorem and the Schauder fixed point theorem. Motivated by the above we will investigate the existence of globally attractivity solutions to the ψ-Hilfer Cauchy fractional problem (1).

This work is organized as follows. In section 2 we present the definitions of the ψ-Riemann-Liouville fractional integral and the ψ-Hilfer fractional derivative and some important results. In section 3 we investigate the globally attractivity existence of solutions of the Cauchy fractional problem.

2 Preliminaries

Let $J=[a,b]$ $(-\infty<a<b<+\infty)$ be a finite interval of $\mathbb{R}$ and Ω a Banach space. The weighted space $\mathcal{C}_{\gamma,\psi}(J,\Omega)$ of functions θ on $(a,b]$ is defined by [6]

$$\mathcal{C}_{\gamma,\psi}(J,\Omega)=\{\theta:(a,b]\to\Omega,(\psi(t)-\psi(a))^{\gamma}\theta(t)\in\mathcal{C}(J,\Omega)\}$$

with $0\leq\gamma<1$ and the norm is given by

$$\begin{aligned}\|\theta\|_{C_{\gamma,\psi}(J,\Omega)} &= \|(\psi(t)-\psi(a))^{\gamma}\theta(t)\|_{C_{\gamma,\psi}(J,\Omega)}\\ &= \max_{t\in J}|(\psi(t)-\psi(a))^{\gamma}\theta(t)|\cdot\end{aligned}$$

The weighted space $\mathcal{C}^{n}_{\gamma,\psi}(J,\Omega)$ of functions θ on $(a,b]$ is defined by

$$\mathcal{C}^{n}_{\gamma,\psi}(J,\Omega)=\{\theta:(a,b]\to\Omega,\theta(t)\in\mathcal{C}^{n-1}(J,\Omega),\theta^{(n)}\in\mathcal{C}_{\gamma,\psi}(J,\Omega)\}$$

with $0\leq\gamma<1$ and the norm is given by

$$\|\theta\|_{C^{n}_{\gamma,\psi}(J,\Omega)}=\sum_{k=0}^{n-1}\left\|\theta^{(k)}\right\|_{\mathcal{C}(J,\Omega)}+\left\|\theta^{(n)}\right\|_{\mathcal{C}_{\gamma,\psi}(J,\Omega)}.$$

For $n=0$, we have $\mathcal{C}^{0}_{\gamma,\psi}(J,\Omega)=\mathcal{C}_{\gamma,\psi}(J,\Omega)$ and

$$\mathcal{C}^{\nu,\eta}_{\gamma,\psi}(J,\Omega)=\left\{\theta\in\mathcal{C}_{\gamma,\psi}(J,\Omega),\ {}^{H}\mathscr{D}^{\nu,\eta;\psi}_{a^+}\theta\in\mathcal{C}_{\gamma,\psi}(J,\Omega)\right\}$$

with $\gamma=\nu+\eta(1-\nu)$.

Let $n-1<\nu<n$, with $n\in\mathbb{N}$, $J=[a,b]$ is an interval such that $-\infty\leq a<b\leq+\infty$ and $\theta,\psi\in\mathcal{C}^{n}(J,\mathbb{R})$ are two functions such that ψ is increasing and $\psi(t)\neq0$, for all $t\in J$. The ψ-Hilfer fractional derivative (left-sided), denoted by ${}^{H}\mathscr{D}^{\nu,\eta;\psi}_{a^+}(\cdot)$ of a function θ of order ν and type $0\leq\eta\leq1$, is defined by [5,8]

$${}^{H}\mathscr{D}^{\nu,\eta;\psi}_{a^+}\theta(t)=\mathcal{I}^{\eta(n-\nu);\psi}_{a^+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}\mathcal{I}^{(1-\eta)(n-\nu);\psi}_{a^+}\theta(t) \tag{2}$$

where $\mathcal{I}^{\nu;\psi}_{a^+}(\cdot)$ is the ψ-Riemann-Liouville fractional integral. Similarly one can define the ψ-Hilfer fractional derivative (right-sided).

Proposition 2.1. [5] Let $\nu > 0$ and $\delta > 0$. If $\theta(t) = (\psi(t) - \psi(0))^{\delta-1}$ then

$$\mathcal{I}_{0^+}^{\nu;\psi}\theta(t) = \frac{\Gamma(\delta)}{\Gamma(\nu+\delta)}(\psi(t) - \psi(0))^{\nu+\delta-1}. \tag{3}$$

Proposition 2.2. [5] Let $\nu > 0$, then

$$^H\mathscr{D}_{0^+}^{\nu,\eta;\psi}(\psi(t) - \psi(0)) = 0 \tag{4}$$

with $^H\mathscr{D}_{0^+}^{\nu,\eta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative.

Assumes that the operator $u : [0, \infty) \times \Omega \to \Omega$ is continuous. The Cauchy fractional problem (1) is equivalent to the integral Volterra equation,

$$\theta(t) = \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}\theta_0 + \frac{1}{\Gamma(\nu)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\nu-1}u(s, \theta(s))\, ds \tag{5}$$

with $t > 0$.

Let $\mathcal{C}^0_{\gamma,\psi}([t_0, \infty), \Omega) = \{\theta \in \mathcal{C}_{\gamma,\psi}([t_0, \infty), \Omega);\ \lim_{t\to\infty} |\theta(t)| = 0\} \cdot$ Note $\mathcal{C}^0_{\gamma,\psi}([0, \infty), \Omega)$ is a Banach space. We need also the following generalized Arzelà-Ascoli theorem [11].

Lemma 2.1. [11] The set $\mathcal{H} \subset \mathcal{C}^0([0, \infty), \Omega)$ is relatively compact if and only if the following conditions hold:
1. for any $T > 0$, the function in $\mathcal{H}$ are equicontinuous on $[0, T]$;
2. for any $t \in [0, \infty)$, $\mathcal{H}(t) = \{\theta(t) : \theta \in \mathcal{H}\}$ is relatively compact in Ω;
3. $\lim\limits_{t\to\infty} |\theta(t)| = 0$ uniformly for $\theta \in \mathcal{H}$.

Lemma 2.2. [4, 11] The noncompact measure $\mu(\cdot)$ satisfies:
1. If for all bounded subsets B_1, B_2 of Ω implies $\mu(B_1) \le \mu(B_2)$;
2. If $\mu(\{x\} \cup B) = \mu(B)$ for every $x \in \Omega$ and every nonempty subset $B \subseteq \Omega$;
3. $\mu(B) = 0$ if and only if B is relatively compact in Ω;
4. $\mu(B_1 + B_2) \le \mu(B_1) + \mu(B_2)$, where $B_1 + B_2 = \{x + y; x \in B_1, y \in B_2\}$;
5. $\mu(B_1 \cup B_2) \le \max\{\mu(B_1), \mu(B_2)\}$;
6. $\mu(\lambda B) \le |\lambda|\, \mu(B)$ for any $\lambda \in \mathbb{R}$.

For any $W \subset C(J, \Omega)$, we define

$$\int_0^t W(s)\, ds = \left\{\int_0^t u(s)\, ds : u \in W\right\}, \text{ for } t \in J. \tag{6}$$

Propriedade 2.1. [4, 11] If $W \subset \mathcal{C}(J, \Omega)$ is bounded and equicontinuous, them $\overline{co}W \subset \mathcal{C}(J, \Omega)$ is also bounded and equicontinuous.

Propriedade 2.2. [4,11] If $W \subset \mathcal{C}(J, \Omega)$ is bounded and equicontinuous, then $t \to \mu(W(t))$ is continuous on J, and

$$\mu(W) = \max_{t\in J}\mu(W(t)),\ \mu\left(\int_0^t W(s)\, ds\right) \le \int_0^t \mu(W(s))\, ds,\ \text{for } t \in J.$$

Propriedade 2.3. [4,11] Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Bochener integrable functions from J in to Ω with $|u_n(t)| \leq \widetilde{m}(t)$ for almost all $t \in J$ and every $n \geq 1$, where $\widetilde{m} \in L(J, \mathbb{R}^+)$, then the function $\widetilde{\Phi}(t) = \mu(\{u_n(t)\}_{n=1}^{\infty})$ belongs to $L(J, \mathbb{R}^+)$ and satisfies

$$\mu\left(\left\{\int_0^t u_n(s)\,ds : n \geq 1\right\}\right) \leq 2\int_0^t \widetilde{\Phi}(s)\,ds.$$

Propriedade 2.4. [4,11] If W is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subset W$, such that

$$\mu(W) \leq \mu(\{u_n(t)\}_{n=1}^{\infty}) + \varepsilon.$$

3 Attractivity with ψ-Hilfer fractional derivative

In this section, we will first discuss two important results, namely, Lemma 3.1 and Lemma 3.2. Then we investigate the existence of attractive solutions of the Cauchy fractional problem.

Now we introduce the following hypothesis:

(C1) $|u(t,\theta)| \leq L(\psi(t)-\psi(0))^{-\xi_1}|\theta|^{\xi_2}$ for $t \in (0,\infty)$ and $\theta \in \Omega$, $L \geq 0$, $\nu < \xi_1 < 1$ and $\xi_2 \in \mathbb{R}$;

(C2) There exists a constant $\overline{k} > 0$ such that for any bounded set $E \subset \Omega$

$$\mu(u(t,E)) \leq \overline{k}\mu(E);$$

here $\mu(\cdot)$ denotes the Hausdorff measure of non compactness.

For all $\theta \in \mathcal{C}_{\gamma,\psi}([0,\infty),\Omega)$ and for a given $n \in \mathbb{N}^+$, we define the operator $\mathcal{T}$ by

$$\mathcal{T}(\theta)(t) = \mathcal{T}_1(\theta)(t) + \mathcal{T}_2(\theta)(t)$$

where

$$\mathcal{T}_1(\theta)(t) = \left[(\psi(t)-\psi(0)) + \frac{1}{n}\right]^{\gamma-1} \frac{\theta_0}{\Gamma(\gamma)} \tag{7}$$

and

$$\mathcal{T}_2(\theta)(t) = \frac{1}{\Gamma(\nu)}\int_0^t \Theta_\psi^{\nu-1}(s,t)\,u(s,\theta(s))\,ds, \tag{8}$$

with $t \in [0,\infty)$.

As $0 < \nu < \xi_2 < 1$, we can choose $\xi > 0$ small enough such that $\nu + \xi - 1 < 0$, $1 - \xi_1 - \xi\xi_2 > 0$ and $\nu + \xi - \xi_1 - \xi\xi_2 < 0$. Note that

$$\begin{aligned}
&|(\mathcal{T}\theta)(t)| \\
\leq\ & \left[(\psi(t)-\psi(0)) + \frac{1}{n}\right]^{\gamma+\xi-1} \frac{|\theta_0|}{\Gamma(\gamma)} + \frac{L}{\Gamma(\nu)}\int_0^t \Theta_\psi^{\nu-1}(s,t)(\psi(s)-\psi(0))^{-\xi_1-\xi\xi_2}\,ds\,.
\end{aligned}$$

Choosing $\delta = \xi\xi_2 - \xi_1 + 1$ and substituting in (3) (Proposition 2.1), yields

$$\mathcal{I}_{0^+}^{\nu,\psi}\theta(t) = \frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t)-\psi(0))^{\nu-\xi_1-\xi\xi_2}.$$

Then, choosing $T > 0$ sufficiently large, from (9), yields

$$\begin{aligned} & |\mathcal{T}(\theta)(t)| \\ \leq & \left[(\psi(t)-\psi(0))+\frac{1}{n}\right]^{\gamma+\xi-1} \frac{|\theta_0|}{\Gamma(\gamma)} + L\frac{\Gamma(1-\xi_1-\xi\xi_2)}{\Gamma(1+\nu-\xi_1-\xi\xi_2)}(\psi(t)-\psi(0))^{\nu-\xi_1-\xi\xi_2+\xi} \\ \leq & \ 1 \end{aligned} \tag{9}$$

for all $t \geq T$.

Define a set $\mathcal{Q}_{\xi;\psi}$ as follows

$$\mathcal{Q}_{\xi;\psi} = \left\{\theta(t)|\theta \in \mathcal{C}_{\gamma,\psi}([0,\infty),\Omega); \left|(\psi(t)-\psi(0))^{\xi}\theta(t)\right| \leq 1,\ t \geq T\right\}. \tag{10}$$

Note $\mathcal{Q}_{\xi;\psi} \neq \emptyset$ and $\mathcal{Q}_{\xi;\psi}$ is a closed convex subset of $\mathcal{C}^0_{\gamma,\psi}([0,\infty),\Omega)$.

Lemma 3.1. [9] Assume (C1) holds. Then, $\{\mathcal{T}\theta; \theta \in \mathcal{Q}_{\xi,\psi}\}$ is equicontinuous and $\lim_{t\to\infty} |\mathcal{T}\theta(t)| = 0$ uniformly for $\theta \in \mathcal{Q}_{\xi,\psi}$.

Lemma 3.2. [9] Suppose (C1) holds. Then, $\mathcal{T}$ takes $\mathcal{Q}_{\xi,\psi}$ into $\mathcal{Q}_{\xi,\psi}$ and is continuous on $\mathcal{Q}_{\xi,\psi}$.

Theorem 3.1. Assume (C1) and (C2) hold. Then, the Cauchy fractional problem (1) admits at least one attractive solution.

Proof. Note $\mathcal{T} : \mathcal{Q}_{\xi,\psi} \to \mathcal{Q}_{\xi,\psi}$ is bounded, continuous (see Lemma 3.2). Also $\{\mathcal{T}\theta : \theta \in \mathcal{Q}_{\xi,\psi}\}$ is equicontinuous and $\lim\limits_{t\to\infty} |\mathcal{T}\theta(t)| = 0$ uniformly for $x \in \mathcal{Q}_{\xi,\psi}$ (see Lemma 3.1), in particular $\{\mathcal{T}_2\theta : \theta \in \mathcal{Q}_{\xi,\psi}\}$.

Let's check that for any $t \in [0,\infty)$, $\{(\mathcal{T}\theta)(t) : \theta \in \mathcal{Q}_{\xi,\psi}\}$ is relatively compact in Ω by using (C2). For each bounded subset $Q_0 \subset \mathcal{Q}_{\xi,\psi}$, set

$$\mathcal{T}^1(Q_0) = \mathcal{T}_2(Q_0),\ \mathcal{T}^n(Q_0) = \mathcal{T}_2\left(\overline{co}\left(\mathcal{T}^{n-1}(Q_0)\right)\right), n = 2,3,...$$

where $\overline{co}$ is closure convex hull [9].

Using the condition $(C2)$, Property 2.4 and Property 2.3, for any $\widetilde{\varepsilon} > 0$, there is a sequence $\left\{\theta_n^{(1)}\right\}_{n=1}^{\infty}$ such that $\mu(\mathcal{T}^{-1}(Q_0(t))) \leq \dfrac{4\overline{k}}{\Gamma(\nu+1)}\mu(Q_0)(\psi(t)-\psi(0))^{\nu} + \widetilde{\varepsilon}$.

Since $\widetilde{\varepsilon} > 0$ is arbitrary, yields

$$\mu\left(\mathcal{T}^1(Q_0(t))\right) \leq \frac{4\overline{k}}{\Gamma(\nu+1)}\mu(Q_0)(\psi(t)-\psi(0))^{\nu}.$$

By means of the Property 2.3 and Property 2.4, for any $\widetilde{\varepsilon} > 0$, there is a sequence $\left\{\theta_n^{(2)}\right\}_{n=1}^{\infty} \subset \overline{co}\left(\mathcal{T}^1(Q_0)\right)$ such that

$$\mu\left(\mathcal{T}^2(Q_0(t))\right) \leq \frac{\left(4\overline{k}\right)^2 \mu(Q_0)}{\Gamma(2\nu+1)}(\psi(t)-\psi(0))^{2\nu} + \widetilde{\varepsilon}.$$

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By mathematical induction, for every $\widetilde{n} \in \mathbb{N}$, yields

$$\mu\left(\mathcal{T}^{\widetilde{n}}\left(Q_0\left(t\right)\right)\right) \leq \frac{\left(4\overline{k}\right)^{\widetilde{n}}\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\nu\widetilde{n}}}{\Gamma\left(\nu\widetilde{n}+1\right)}\mu\left(Q_0\right).$$

Since

$$\lim_{\widetilde{n}\to\infty}\frac{\left[4\overline{k}\left(\psi\left(a\right)-\psi\left(0\right)\right)^{\nu}\right]^{\widetilde{n}}}{\Gamma\left(\nu\widetilde{n}+1\right)} = 0,$$

there exists $m \in \mathbb{Z}_+$ such that

$$\frac{\left(4\overline{k}\right)^{m}\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\nu m}}{\Gamma\left(\nu m+1\right)} \leq \frac{\left[4\overline{k}\left(\psi\left(a\right)-\psi\left(0\right)\right)^{\nu}\right]^{m}}{\Gamma\left(\nu m+1\right)} = \widetilde{q} < 1.$$

Then

$$\mu\left(\mathcal{T}^{m}\left(Q_0\left(t\right)\right)\right) \leq \widetilde{q}\mu\left(Q_0\right).$$

We know from Property 2.1, $\mathcal{T}^{m}\left(Q_0\left(t\right)\right)$ is bounded and equicontinuous. Then, by Property 2.2, we get $\mu\left(\mathcal{T}^{m}\left(Q_0\right)\right) = \max_{t\in[0,a]}\mu\left(\mathcal{T}^{m}\left(Q_0\left(t\right)\right)\right)$. Hence $\mu\left(\mathcal{T}^{m}\left(Q_0\right)\right) \leq \widetilde{q}\mu\left(Q_0\right)$.

There exist $\exists\widetilde{D} \subset Q_{\xi,\psi}$, such that $\mu\left(\mathcal{T}_2\left(\widetilde{D}\right)\right) = 0$, i.e., $\mathcal{T}_2\left(\widetilde{D}\right)$ is relatively compact.

As $\{(\mathcal{T}\theta)(t) : \theta \in \mathcal{Q}_{\xi,\psi}\}$ is relatively compact for any $t \in [0,\infty)$, then, every sequence $\{\theta_n\}$ in $\mathcal{Q}_{\xi,\psi}$ admit a uniformly convergent subsequence $\{\theta_{n_k}\}$ in $C^0_{\gamma,\psi}(J,\Omega)$ ($\mathcal{Q}_{\xi,\psi} \subset C^0_{\gamma,\psi}(J,\Omega)$) by Arzelà-Ascoli theorem.

Furthermore, $\{\theta_{n_k}\}$ satisfies

$$\theta_{n_k}(t) = \left[(\psi(t)-\psi(0)+\frac{1}{n_k}\right]^{\gamma-1}\frac{\theta_0}{\Gamma(\gamma)} + \frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}\left(s,t\right)u(s,\theta_{n_k}(s))\,\mathrm{d}s, \tag{11}$$

with $t \in [0,\infty)$.

Let $\theta^*(t) = \lim_{k\to\infty}\theta_{n_k}(t)$ $(t \neq 0)$. The Lebesgue dominated convergence theorem with (11) yields

$$\theta^*(t) = \left[(\psi(t)-\psi(0)+\frac{1}{n}\right]^{\gamma-1}\frac{\theta_0}{\Gamma(\gamma)} + \frac{1}{\Gamma(\nu)}\int_0^t \Theta_{\psi}^{\nu-1}\left(s,t\right)u(s,\theta^*(s))\,ds,$$

with $t \in [0,\infty)$, so $\theta^*(t)$ is an attractive solution for the Cauchy fractional problem. □

Corollary 3.1. [9] Assume (C1) holds. Then, the Cauchy fractional problem (1) admits at least one attractive solution.

Conclusion

In this work, we discuss the attractivity of solutions for a system of fractional differential equations in the sense of the ψ-Hilfer fractional derivative, using Hausdorff's non-compactness measure and the Arzelà-Ascoli.

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