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A topological derivative-based method for the reconstruction of multiple pollution sources

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Abstract. The topological derivative method is used to solve a pollution sources reconstruction problem governed by a steady-state convection-diffusion equation. The inverse problem consists in the reconstruction of a set of pollution sources in a fluid medium by measuring the concentration of the pollutants within some subregion of the reference domain. We rewrite the inverse problem as a topology optimization problem which allows us to solve it by using the concept of topological derivatives. The resulting algorithm is able to reconstruct the pollution sources in one step and is independent of any initial guess. A numerical example is presented to show the effectiveness of our reconstruction method.

Keywords. Inverse problem, pollution sources reconstruction, topological derivative method.

1 Introduction

In this paper, we deal with an inverse reconstruction problem in $\mathbb{R}^2$ whose corresponding forward problem is governed by a steady-state convection-diffusion equation. The inverse problem under consideration is about the reconstruction of a set of pollution sources in a fluid medium by measuring the concentration of the pollutants within a subregion of the domain. By assuming the velocity of the leakages is given, we reconstruct the topology of the pollution sources by recovering their locations and sizes. More precisely, let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain with smooth boundary $\partial\Omega$. Measurements of the pollutant concentration field are collected in a subdomain $\Omega_o \subset \Omega$. We assume that there may be an unknown number ($N^* \in \mathbb{Z}^+$) of isolated pollution sources ω_i^* within the domain Ω. Therefore, there is a set $\omega^* = \cup_{i=1}^{N^*}\omega_i^*$, whose components ω_i^* satisfy $\overline{\omega_i^*} \cap \overline{\omega_j^*} = \emptyset$ for $i \neq j$ and $\overline{\omega_i^*} \cap \partial\Omega = \emptyset$ for each $i, j \in \{1, \dots, N^*\}$.

We assume that the polluting substance is leaking in an incompressible flow with velocity V, such that $V \neq (0,0)$ at each $\omega_i^* \subset \Omega$, $i = 1, \dots, N^*$. For a given Dirichlet data g imposed on $\partial\Omega$, the resulting pollutant concentration z in Ω is observed in the subdomain Ω_o. In this set up, the inverse problem consists in finding χ_{ω^*} such that the pollutant concentration z satisfies the following boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta z + \chi_{\omega^*} V \cdot \nabla z &=& 0 & \text{in } \Omega, \\ z &=& g & \text{on } \partial\Omega, \end{array} \right. \quad \text{with} \quad \chi_{\omega^*} = \left\{ \begin{array}{ll} 0 & \text{in } \Omega \setminus \overline{\omega^*}, \\ 1 & \text{in } \omega_i^*,\ i = 1, \dots, N^* \end{array} \right. \tag{1}$$

and the velocity V is considered as a null divergence vector field, namely, $\operatorname{div} V = 0$ in ω_i^*, for $i = 1, \dots, N^*$. Here, function g is a purely synthetic data used to produce the observation of z in Ω_o. The realistic problem of pollution source reconstruction is much more complicated and requires

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further investigation. Now, for an initial guess χ_ω of χ_{ω^*}, we consider the pollutant concentration field u to be the solution to the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u + \chi_\omega V \cdot \nabla u & = & 0 & \text{in} \quad \Omega, \\ u & = & g & \text{on} \quad \partial\Omega, \end{array} \right. \quad \text{with} \quad \chi_\omega = \left\{ \begin{array}{ll} 0 & \text{in } \Omega \setminus \overline{\omega}, \\ 1 & \text{in } \omega_i, \ i = 1, \ldots, N. \end{array} \right. \tag{2}$$

The characteristic function χ_{ω^*} is unknown and hence z but we assume that z can be measured in Ω_o. Thus, if we look for an appropriate χ_{ω^*}, then we wish u to agree with z in Ω_o, i.e., we want $u = z_{|\Omega_o}$. Keeping this objective in mind, we rewrite the inverse problem in the form of the following topology optimization problem given by

$$\underset{\omega\subset\Omega}{\text{Minimize}} \ \mathcal{J}_\omega(u^1, \ldots, u^M) = \sum_{m=1}^{M} \int_{\Omega_o} (u^m - z^m)^2, \tag{3}$$

where $M \in \mathbb{Z}^+$ is the number of observations, z^m and u^m are the solutions of the boundary value problems (1) and (2), respectively, corresponding to the Dirichlet data g^m with $m = 1, \ldots, M$. In particular, problem (3) is minimized with respect to a set of ball-shaped pollution sources by using the concept of topological derivatives [4]. It means that the shape functional $\mathcal{J}_\omega(u^1, \ldots, u^M)$ is expanded asymptotically and then truncated up to the second-order term. The resulting expression is trivially minimized with respect to the parameters under consideration, leading us to a noniterative second-order reconstruction algorithm.

The remaining part of this paper is organized as follows. Since the proposed inverse problem is to be solved by using the concept of topological derivatives, we present in Section 2 the shape functionals corresponding to the unperturbed and perturbed domains as well as the *ansatz* for the scalar field associated to the topologically perturbed domain. In Section 3, the asymptotic expansion of the shape functional is presented. Based on such asymptotic expansion of the shape functional, a noniterative reconstruction algorithm is devised in Section 4 and a numerical example is presented in Section 5. Concluding remarks are discussed in Section 6.

2 Topology optimization setting

The inverse problem of finding χ_{ω^*} has been written in the form of the optimization problem (3). It is well known that a quite general approach for dealing with such class of problems is based on the concept of topological derivative, which consists in expanding the shape functional $\mathcal{J}_\omega(u^1, \ldots, u^M)$ with respect to the parameters depend upon a set of small perturbations. The reader may refer to the book of Novotny & Sokołowski [1] to be familiar with the notion of topological derivatives. Since the topological derivative does not depend on the initial guess of the unknown topology ω^*, we start with the unperturbed domain by setting $\omega = \emptyset$. More precisely, we consider

$$\mathcal{J}(u_0^1, \ldots, u_0^M) = \sum_{m=1}^{M} \int_{\Omega_o} (u_0^m - z^m)^2 \, , \tag{4}$$

where u_0^m be the solution of the unperturbed boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u_0^m & = & 0 & \text{in } \Omega, \\ u_0^m & = & g^m & \text{on } \partial\Omega. \end{array} \right. \tag{5}$$

Here, we are considering the topology optimization problem (3) for the ball-shaped pollution sources and hence we define the topologically perturbed counter-part of (5) by introducing $N \in \mathbb{Z}^+$ number of small circular pollution sources $B_{\varepsilon_i}(x_i)$ with center at $x_i \in \Omega$ and radius ε_i

for $i = 1, \ldots, N$. The set of circular pollution sources is denoted as $B_\varepsilon(\xi) = \cup_{i=1}^N B_{\varepsilon_i}(x_i)$, where $\xi = (x_1, \ldots, x_N)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$. Moreover, we assume that $\overline{B_\varepsilon} \cap \partial\Omega = \emptyset$, $\overline{B_\varepsilon} \cap \Omega_o = \emptyset$ and $\overline{B_{\varepsilon_i}(x_i)} \cap \overline{B_{\varepsilon_j}(x_j)} = \emptyset$ for each $i \neq j$ and $i, j \in \{1, \ldots, N\}$. The shape functional associated with the topologically perturbed domain is written as

$$\mathcal{J}(u_\varepsilon^1, \ldots, u_\varepsilon^M) = \sum_{m=1}^M \int_{\Omega_o} (u_\varepsilon^m - z^m)^2 \tag{6}$$

with u_ε^m be the solution of the perturbed boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u_\varepsilon^m + \chi_\varepsilon V \cdot \nabla u_\varepsilon^m &=& 0 & \text{in } \Omega, \\ u_\varepsilon^m &=& g^m & \text{on } \partial\Omega, \end{array} \right. \quad \text{with} \quad \chi_\varepsilon = \left\{ \begin{array}{ll} 0 & \text{in } \Omega \setminus \overline{B_\varepsilon(\xi)}, \\ 1 & \text{in } B_\varepsilon(\xi). \end{array} \right. \tag{7}$$

By construction, we consider that the velocity field V satisfies the additional condition $\operatorname{div} V = 0$ in $B_\varepsilon(\xi)$ and in order to simplify the analysis we assume that the velocity V is constant in the neighborhood of $B_{\varepsilon_i}(x_i)$, so that $V(x) \approx V_i := V(x_i)$, with $x \in B_{\varepsilon_i}(x_i)$, $i = 1, \ldots, N$.

As an initial step to obtain the topological derivatives of the shape functional $\mathcal{J}$ at u_ε, we start by simplifying the difference between the perturbed shape functional $\mathcal{J}(u_\varepsilon^1, \ldots, u_\varepsilon^M)$ and its unperturbed counter-part $\mathcal{J}(u_0^1, \ldots, u_0^M)$ defined in (6) and (4), respectively, as follows

$$\mathcal{J}(u_\varepsilon) - \mathcal{J}(u_0) = \sum_{m=1}^M \int_{\Omega_o} \left[2(u_\varepsilon^m - u_0^m)(u_0^m - z^m) + (u_\varepsilon^m - u_0^m)^2 \right], \tag{8}$$

where $u_\varepsilon = (u_\varepsilon^1, \ldots, u_\varepsilon^M)$ and $u_0 = (u_0^1, \ldots, u_0^M)$. Moreover, we also introduce the vector $\alpha \in \mathbb{R}^N$ where each entry α_i denotes the Lebesgue measure (volume) of the two-dimensional ball $B_{\varepsilon_i}(x_i)$, i.e.,

$$\alpha = (\alpha_1, \ldots, \alpha_N) \quad \text{with} \quad \alpha_i = |B_{\varepsilon_i}(x_i)| = \pi\varepsilon_i^2, \quad \text{for} \quad i = 1, \ldots, N, \tag{9}$$

and the characteristic function $\chi_i := \chi_{B_{\varepsilon_i}(x_i)}$. Now, we propose the following *ansatz* for the expansion of u_ε^m with respect to the parameters corresponding to the circular pollution sources in the form

$$u_\varepsilon^m(x) = u_0^m(x) + \sum_{i=1}^N \alpha_i h_i^{\varepsilon,m}(x) + \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j h_{ij}^{\varepsilon,m}(x) + \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j^2 \varphi_{ij}^{\varepsilon,m}(x) + \tilde{u}_\varepsilon^m(x), \tag{10}$$

where, for each $i, j = 1, \ldots, N$ and $m = 1, \ldots, M$, the functions $h_i^{\varepsilon,m}$, $h_{ij}^{\varepsilon,m}$, $\varphi_{ij}^{\varepsilon,m}$ and $\tilde{u}_\varepsilon^m$ are the solutions of

$$\left\{ \begin{array}{ll} \Delta h_i^{\varepsilon,m} = \alpha_i^{-1} V_i \cdot \nabla u_0^m \, \chi_i & \text{in } \Omega, \\ h_i^{\varepsilon,m} = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{11}$$

$$\left\{ \begin{array}{ll} \Delta h_{ij}^{\varepsilon,m} = \alpha_i^{-1} V_i \cdot \nabla h_j^{\varepsilon,m} \, \chi_i & \text{in } \Omega, \\ h_{ij}^{\varepsilon,m} = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{12}$$

$$\left\{ \begin{array}{ll} \Delta \varphi_{ij}^{\varepsilon,m} = \alpha_i^{-1} V_i \cdot \nabla h_{jj}^{\varepsilon,m} \, \chi_i & \text{in } \Omega, \\ \varphi_{ij}^{\varepsilon,m} = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{13}$$

$$\left\{ \begin{array}{ll} -\Delta \tilde{u}_\varepsilon^m + \chi_\varepsilon V \cdot \nabla \tilde{u}_\varepsilon^m = \Phi_\varepsilon^m & \text{in } \Omega, \\ \tilde{u}_\varepsilon^m = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{14}$$

respectively. In problem (14), we have $\Phi_\varepsilon^m = \Phi_{\varepsilon,1}^m + \Phi_{\varepsilon,2}^m$, with

$$\Phi_{\varepsilon,1}^m = -\sum_{i=1}^N \sum_{j=1}^N \sum_{\substack{l=1 \\ l \neq j}}^N \alpha_j \alpha_l V_i \cdot \nabla h_{jl}^{\varepsilon,m} \, \chi_i \quad \text{and} \quad \Phi_{\varepsilon,2}^m = -\sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N \alpha_j \alpha_l^2 V_i \cdot \nabla \varphi_{jl}^{\varepsilon,m} \, \chi_i. \tag{15}$$

In view of capturing the behavior of the functions $h_i^{\varepsilon,m}$ and $h_{ij}^{\varepsilon,m}$ with respect to the small parameter ε, we propose a decomposition of them in five and three functions, respectively. Firstly, we rewrite $h_i^{\varepsilon,m}$ as

$$h_i^{\varepsilon,m} = (V_i \cdot \nabla u_0^m(x_i))(p_i^\varepsilon + q_i) + \tilde{h}_i^{\varepsilon,m}. \tag{16}$$

For a positive real number $R \gg \varepsilon$, we consider the ball $B_R(x_i)$ such that $B_{\varepsilon_i}(x_i) \subset \Omega \subset B_R(x_i)$ with $x_i \in \Omega$ for $i = 1, \ldots, N$. Then, the function p_i^ε satisfies the problem

$$\left\{ \begin{array}{rcll} \Delta p_i^\varepsilon & = & \alpha_i^{-1}\chi_i & \text{in } B_R(x_i), \\ p_i^\varepsilon & = & (2\pi)^{-1}\ln R & \text{on } \partial B_R(x_i), \end{array} \right. \tag{17}$$

whose solution is given by

$$p_i^\varepsilon = \left\{ \begin{array}{ll} \dfrac{1}{4\pi\varepsilon_i^2}\|x - x_i\|^2 - \dfrac{1}{2\pi}\left(\dfrac{1}{2} - \ln \varepsilon_i\right) & \text{in } B_{\varepsilon_i}(x_i), \\ \dfrac{1}{2\pi}\ln\|x - x_i\| & \text{in } B_R(x_i) \setminus \overline{B_{\varepsilon_i}(x_i)}. \end{array} \right. \tag{18}$$

From (18), one can observe that the solution p_i^ε does not depend on ε_i outside the ball $B_{\varepsilon_i}(x_i)$. Therefore, we use the notation $p_i(x) := p_i^\varepsilon(x)$, $\forall x \in \Omega \setminus \overline{B_{\varepsilon_i}(x_i)}$. Additionally, q_i is the solution to the homogeneous boundary value problem

$$\left\{ \begin{array}{rcll} \Delta q_i & = & 0 & \text{in } \Omega, \\ q_i & = & -p_i & \text{on } \partial\Omega. \end{array} \right. \tag{19}$$

Next, by introducing the vector quantity $W_i^m := (\nabla^2 u_0^m)V_i$, for $m = 1, \ldots, M$, function $\tilde{h}_i^{\varepsilon,m}$ can be decomposed as

$$\tilde{h}_i^{\varepsilon,m} := \tilde{p}_i^{\varepsilon,m} + \varepsilon_i^2 \tilde{q}_i^m + \tilde{\tilde{h}}_i^{\varepsilon,m}, \tag{20}$$

where $\tilde{p}_i^{\varepsilon,m}$ is solution to the problem

$$\left\{ \begin{array}{rcll} \Delta \tilde{p}_i^{\varepsilon,m} & = & \alpha_i^{-1} W_i^m \cdot (x - x_i)\chi_i & \text{in } B_R(x_i), \\ \tilde{p}_i^{\varepsilon,m} & = & 0 & \text{on } \partial B_R(x_i), \end{array} \right. \tag{21}$$

which admits the explicit solution

$$\tilde{p}_i^{\varepsilon,m} = \left\{ \begin{array}{ll} \dfrac{1}{8\pi}\left[\dfrac{\|x - x_i\|^2 - \varepsilon_i^2}{\varepsilon_i^2} - \dfrac{R^2 - \varepsilon_i^2}{R^2}\right] W_i^m \cdot (x - x_i) & \text{in } B_{\varepsilon_i}(x_i), \\ \dfrac{\varepsilon_i^2}{8\pi R^2}\dfrac{\|x - x_i\|^2 - R^2}{\|x - x_i\|^2} W_i^m \cdot (x - x_i) & \text{in } B_R(x_i) \setminus \overline{B_{\varepsilon_i}(x_i)}. \end{array} \right. \tag{22}$$

The next term in decomposition (20), given by $\tilde{q}_i^m$, is solution of the following boundary value problem

$$\left\{ \begin{array}{rcll} \Delta \tilde{q}_i^m & = & 0 & \text{in } \Omega, \\ \tilde{q}_i^m & = & -\tilde{p}_i^m & \text{on } \partial\Omega, \end{array} \right. \tag{23}$$

where $\tilde{p}_i^m := \varepsilon_i^{-2}\tilde{p}_i^{\varepsilon,m}$ in $B_R(x_i) \setminus \overline{B_{\varepsilon_i}(x_i)}$ and hence the independence of $\tilde{q}_i^m$ with respect to ε_i is related to the explicit solution $\tilde{p}_i^{\varepsilon,m}$ of (22).

Similarly to the decomposition proposed to the function $h_i^{\varepsilon,m}$ in (16), we write the function $h_{ii}^{\varepsilon,m}$ in the form

$$h_{ii}^{\varepsilon,m} = (V_i \cdot \nabla u_0^m(x_i))(p_{ii}^\varepsilon + q_{ii}) + \tilde{h}_{ii}^{\varepsilon,m}. \tag{24}$$

The function p_{ii}^ε is solution to the problem

$$\left\{ \begin{array}{rcll} \Delta p_{ii}^\varepsilon & = & \alpha_i^{-1} V_i \cdot \nabla p_i^\varepsilon \chi_i & \text{in } B_R(x_i), \\ p_{ii}^\varepsilon & = & 0 & \text{on } \partial B_R(x_i). \end{array} \right. \tag{25}$$

Problem (25) can be solved analytically and its solution is given by

$$p_{ii}^{\varepsilon} = \begin{cases} \dfrac{1}{(4\pi\varepsilon_i)^2}\left[\dfrac{\|x-x_i\|^2-\varepsilon_i^2}{\varepsilon_i^2} - \dfrac{R^2-\varepsilon_i^2}{R^2}\right] V_i\cdot(x-x_i) & \text{in } B_{\varepsilon_i}(x_i), \\ \dfrac{1}{(4\pi R)^2}\dfrac{\|x-x_i\|^2-R^2}{\|x-x_i\|^2}\, V_i\cdot(x-x_i) & \text{in } B_R(x_i)\setminus\overline{B_{\varepsilon_i}(x_i)}. \end{cases} \tag{26}$$

From (26), one can observe that the solution p_{ii}^{ε} does not depend on ε_i outside the ball $B_{\varepsilon_i}(x_i)$. Therefore, we use the notation $p_{ii}(x) := p_{ii}^{\varepsilon}(x)$, $\forall x \in \Omega\setminus\overline{B_{\varepsilon_i}(x_i)}$. The function q_{ii} is solution to the following boundary value problems

$$\begin{cases} \Delta q_{ii} &= 0 & \text{in } \Omega, \\ q_{ii} &= -p_{ii} & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Since the functions $\tilde{\tilde{h}}_i^{\varepsilon,m}$ and $\tilde{h}_{ii}^{\varepsilon,m}$ do not contribute to the closed form of the topological derivatives, we opted by omitting the problems associated with them from the text. Finally, in order to simplify further calculations related to the asymptotic development of the topologically perturbed shape functional (6), we introduce an adjoint state v^m to be the solution to the problem

$$\begin{cases} -\Delta v^m &= (u_0^m - z^m)\chi_{\Omega_o} & \text{in } \Omega, \\ v^m &= 0 & \text{on } \partial\Omega. \end{cases} \tag{28}$$

3 Topological asymptotic expansion

Now, we have all the elements to evaluate the difference (8) explicitly. We proceed accordingly to the following sequence of steps: (i) the scalar field u_ε^m in (8) is replaced by the expression given by the *ansatz* introduced in (10); (ii) the weak formulations of the problems (11), (12), (13) and (28) are used in order to rewrite the integrals defined over the subdomain Ω_o by integrals defined over the ball $B_{\varepsilon_i}(x_i)$; and (iii) the decompositions for the functions $h_i^{\varepsilon,m}$, $\tilde{h}_i^{\varepsilon,m}$ and $h_{ii}^{\varepsilon,m}$ are then used taking into account the analytical solutions for p_i^{ε}, $\tilde{p}_i^{\varepsilon,m}$ and $p_{ii}^{\varepsilon,m}$, given by (18), (22) and (26), respectively, together with Taylor's expansions for the functions u_0^m, v^m and q_i around the point x_i (the center of the ball B_{ε_i}). As a result, we obtain the asymptotic expansion of the shape functional $\mathcal{J}_\omega(u^1,\ldots,u^M)$ in its matrix form given by

$$\psi(\chi_\varepsilon(\xi)) = \psi(\chi) - \alpha\cdot d(\xi) + \frac{1}{2}H(\xi)\alpha\cdot\alpha + o(|\alpha|^2)\ , \tag{29}$$

where $\psi(\chi_\varepsilon(\xi)) := \mathcal{J}(u_\varepsilon)$ and $\psi(\chi) := \mathcal{J}(u_0)$. The entries of the vector $d\in\mathbb{R}^N$ are defined as

$$d_i := -2\sum_{m=1}^{M}(V_i\cdot\nabla u_0^m(x_i))v^m(x_i), \tag{30}$$

and, by introducing the quantity $h_i := p_i + q_i$, the entries of the matrix $H\in\mathbb{R}^{N\times N}$ are such that

$$\begin{aligned} H_{ii} := &-\frac{1}{\pi}\sum_{m=1}^{M} V_i\cdot\nabla^2 u_0^m(x_i)\nabla v^m(x_i) + \frac{1}{2\pi}\sum_{m=1}^{M}(V_i\cdot(\nabla^2 u_0^m(x_i)\ V_i))v^m(x_i) \\ &- \frac{1}{2\pi}\sum_{m=1}^{M}(V_i\cdot\nabla u_0^m(x_i))(V_i\cdot\nabla v^m(x_i)) - 4\sum_{m=1}^{M}(V_i\cdot\nabla u_0^m(x_i))(V_i\cdot\nabla q_i(x_i))v^m(x_i) \\ &+ \frac{1}{4\pi}\sum_{m=1}^{M}(V_i\cdot\nabla u_0^m(x_i))\|V_i\|^2 v^m(x_i) + 2\sum_{m=1}^{M}(V_i\cdot\nabla u_0^m(x_i))^2\int_{\Omega_o} h_i^2, \end{aligned} \tag{31}$$

$$H_{ij} := -4\sum_{m=1}^{M}(V_j\cdot\nabla u_0^m(x_j))(V_i\cdot\nabla h_j(x_i))v^m(x_i) + 2\sum_{m=1}^{M}(V_i\cdot\nabla u_0^m(x_i))(V_j\cdot\nabla u_0^m(x_j))\int_{\Omega_o} h_i h_j, \quad (32)$$

if $i \neq j$, respectively, for $i, j = 1, \ldots, N$.

4 Reconstruction algorithm

The expression on the right-hand side of (29) depends on the number of pollution sources N, their sizes α and locations ξ. Thus, by desregarding the terms of order $o(|\alpha|^2)$ from (29), we define $\delta J(\alpha, \xi, N) := -\alpha \cdot d(\xi) + 1/2H(\xi)\alpha \cdot \alpha$. The derivative of the function $\delta J(\alpha, \xi, N)$ with respect to the variable α yields the first-order optimality condition, namely, $\langle D_\alpha \delta J, \beta\rangle = 0, \forall \beta \in \mathbb{R}^N$, which leads to the linear system

$$H(\xi)\alpha = d(\xi) \quad (33)$$

with the entries of the vector $d \in \mathbb{R}^N$ and the matrix $H \in \mathbb{R}^{N\times N}$ defined in (30) and (31)-(32), respectively.

The quantity α solution of (33) becomes a function of the locations ξ, namely $\alpha = \alpha(\xi)$, and its value is obtained by $\alpha = H^{-1}(\xi)d(\xi)$. Now, let us replace the solution of (33) into $\delta J(\alpha, \xi, N)$ to obtain $\delta J(\alpha(\xi), \xi, N) = -d(\xi) \cdot \alpha(\xi)/2$. Therefore, the pair of vectors $(\xi^\star, \alpha^\star)$ which minimizes $\delta J(\alpha, \xi, N)$ is given by $\xi^\star = \operatorname{argmin}_{\xi\in X}\delta J(\alpha(\xi), \xi, N)$ and $\alpha^\star := \alpha(\xi^\star)$, where X is the set of admissible locations for pollution sources. In other words, the minimizer of $\delta J(\alpha, \xi, N)$ is a set of ball-shaped disjoint inclusions which is completely characterized by the pair $(\xi^\star, \alpha^\star)$. In short, for a given number of pollution sources N we want to reconstruct, our method is able to find in one step their sizes $\alpha^\star$ and locations $\xi^\star$. For more sophisticated approaches based on metaheuristic and multi-grid methods, we refer to [3], where the algorithm proposed in this section can be found in pseudo-code format. For more applications of this algorithm, see [2] (Chapter 10), for instance.

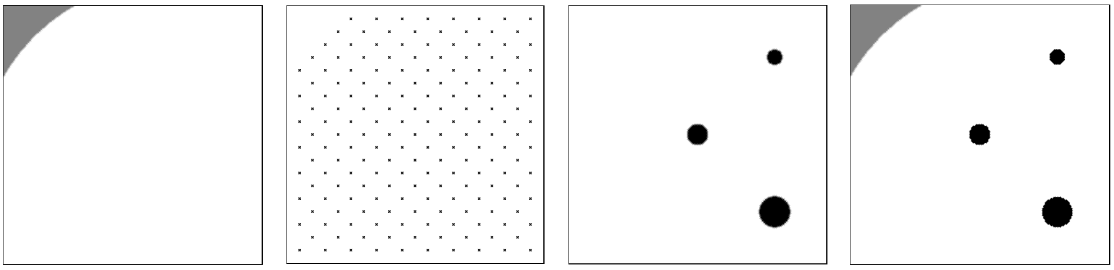
5 Numerical example

A numerical example is presented here to demonstrate the effectiveness of the method proposed in the earlier sections of this paper.

The reference domain is taken as a square $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$ which is discretized with three-node finite elements. The mesh is generated from a grid of 160×160 squares. Each square is divided into four triangles which leads us to a resulting mesh comprising 102400 elements and 51521 nodes. Measurements of scalar fields of interest are taken in the subdomain Ω_o which is the region concentrated in the neighborhood of the left top corner of the square domain Ω. See Figure 1(a). The boundary $\partial\Omega$ of the reference domain is excited by imposing only one Dirichlet data, namely, $g = x$. The auxiliary boundary value problems are solved over the resulting mesh. However, due to the high complexity of the algorithm [3], a subgrid X of uniformly distributed points is extracted from the original mesh and then used as a set of admissible locations where a combinatorial search is performed which leads to the optimal solution $(\xi^\star, \alpha^\star)$ defined in X. In the example, we consider a subgrid X comprising 177 points, as illustrated in Figure **??**. In the figures below, we represented the pollution sources ω^* by black, the subdomain Ω_o where the information is collected by gray and the remaining domain $\Omega \setminus \Omega_o$ by white color.

The example: A pipeline network within the reference domain contains three leakages which are located at the points $x_1^* = (0.3, 0.3)$, $x_2^* = (0, 0)$ and $x_3^* = (0.3, -0.3)$ and their sizes are $\varepsilon_1^* = 0.03$,

$\varepsilon_2^* = 0.04$ and $\varepsilon_3^* = 0.06$. We illustrate the target domain in Figure 1(c). The pollution substance escapes from the pipeline network with a constant velocity given by $V = (2, -1)$. From only one measurement with $g = x$, the algorithm successfully find the exact locations of the pollution sources, i.e., $x_i^\star = x_i^*$, for $i = 1, 2, 3$. The radii obtained were $\varepsilon_1^\star = 0.0301$, $\varepsilon_2^\star = 0.0406$ and $\varepsilon_3^\star = 0.0589$ which are approximately equal to the true values.



(a) (b) (c) (d)

Figure 1: (a) Subdomain Ω_o. (b) Subgrid X. (c) Target domain. (d) Reconstruction from one measurement with $g = x$.

6 Conclusions

In this paper, a noniterative algorithm to solve a pollution sources reconstruction problem is proposed. The inverse problem here addressed is rewrite as a topology optimization problem which allows us to solve it through a method based on higher-order topological derivatives. For a given velocity field V and number of trial pollution sources N, the algorithm devised is able to find their locations and sizes in one step. Moreover, such an algorithm is independent of any initial guess. On the other hand, the proposed method approximates the unknown set of pollution sources by several balls which can be seen as a limitation of our approach. However, it can be used to get a good initial guess for more sophisticated iterative approaches based on level-sets methods, for instance.

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