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Using symbolic expressions to get the Taylor expansion of the Lagrangian auto-covariance function

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Abstract. This work describes how we derived symbolic expressions for the Taylor coefficients of the Lagrangian auto-covariance function for a two-dimensional, mean-zero, homogeneous, steady, and incompressible random velocity field which is written as a sum of many Fourier modes. Additionally, we wrote code to calculate these coefficients exactly and produce executable code which quickly calculates the first terms of the Taylor expansion of the Lagrangian auto-covariance function.

Key words. Velocity field, Lagrangian auto-covariance, Taylor expansion, passive tracer transport.

1 Introduction

The passive tracer transport problem is an important topic from the statistical fluid mechanics field. It consists of determining the probability law of the position $\mathbf{X}_t, t \geq 0$ of one single particle at time $t \geq 0$ which is moved by a random velocity field $\mathbf{U}$. For references, see [2, 6, 9].

Let $\mathbf{U} = \{\mathbf{U}(\mathbf{x}, t), \mathbf{x} \in \mathbb{R}^2, t \geq 0\}$ be a random velocity field taking values in $\mathbb{R}^2$ and let $\mathbf{X}_t$ be the particle position at time t, for $t \geq 0$. So $\{\mathbf{X}_t, t \geq 0\}$ is the solution of the differential equation

$$\frac{d\mathbf{X}_t}{dt} = \mathbf{U}(\mathbf{X}_t, t), \quad t > 0; \quad \mathbf{X}_0 = \mathbf{0}. \tag{1}$$

In general, the main goal consists in determining the law of the entire stochastic location process $\mathbf{X} = \{\mathbf{X}_t, t \geq 0\}$, given the law of the random velocity field $\mathbf{U}$. Despite much work being done on this problem, we still have only a limited ability to derive results about the law of the particle position $\mathbf{X}_t$ from the law of the velocity field $\mathbf{U}(\mathbf{x}, t)$.

Closely related to the passive tracer problem, there exists the problem of determining the law of the Lagrangian velocity process $\mathbf{U} = \{\mathbf{U}(\mathbf{X}_t, t), t \geq 0\}$, which is the particle's velocity viewed by an observer whose location $\mathbf{X}_t$ is determined by the environment. The Eulerian description provided by $\mathbf{U}(\mathbf{x}, t)$, for which the coordinate system is fixed, is different from the Lagrangian description $\mathbf{U}(\mathbf{X}_t, t)$, that gives a description of the velocity field from the view of a particle following the velocity field. Naturally, $\mathbf{X}_t$ is the time integral of $\mathbf{U}(\mathbf{X}_s, s)$ from 0 to t.

For this work we use a velocity field written as a sum of finitely many Fourier modes as

$$\mathbf{U}(\mathbf{x}, t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} R_n \sin(\mathbf{W}_n \cdot \mathbf{x} + \Phi_n)\, \mathbf{\Theta}_n, \quad \mathbf{x} \in \mathbb{R}^2, \tag{2}$$

with $\mathbf{\Theta}_n = \mathbf{W}_n^{\perp} = [-W_{n,2}, W_{n,1}]^T$, where $\mathbf{W}_n = [W_{n,1}, W_{n,2}]^T$, for $n = 1, 2, \ldots, N$, and random amplitudes R_n and random wave numbers $\mathbf{W}_n$ are independent of the random phases Φ_n, in the

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sense that the collection $(R_1, \mathbf{W}_1, R_2, \mathbf{W}_2, \cdots, R_N, \mathbf{W}_N)$ is independent of $(\Phi_1, \Phi_2, \ldots, \Phi_N)$. In addition, we assume that the random phases Φ_n, $n = 1, 2, \ldots, N$, are independent and uniformly distributed on $[0, 2\pi]$, and random vectors $(R_n, \mathbf{W}_n)$, $n = 1, 2, \ldots, N$ have finite joint moments.

The velocity field $\mathbf{U}(\mathbf{x}, t)$ is two-dimensional, mean-zero, homogeneous, stationary, and incompressible [7]; these are key statistical properties of real turbulence. Unlike real turbulence, however, it is also steady; $\mathbf{U}(\mathbf{x}, t)$ does not actually depend on t. However, by choosing the amplitudes R_n and wave numbers $\mathbf{W}_n$, one can approximate a wide variety of probability laws for $\mathbf{U}$. The wave numbers $\mathbf{W}_n$ can take on large values to model rapidly-changing flow fields. The velocity field is not periodic, so it can model an arbitrarily large flow field with large–scale changes driven by small wave numbers $\mathbf{W}_n$. Finally, as we will see below, the simplest form of the results occurs as N goes to infinity, so that a large variety of wave numbers is included in the model.

Similar models for the random velocity field $\mathbf{U}(\mathbf{x}, t)$ are available in the literature and are used combined with computer simulations to get many numerical results [1, 3–5]. Different from many works, we perform exact calculations of important statistics of the Lagrangian velocity $\mathbf{U}(\mathbf{X}_t, t)$ and, ultimately, obtain information about the stochastic process $\{\mathbf{X}_t, t \geq 0\}$ without using simulations or approximations.

2 The Lagrangian auto-covariance function

The Lagrangian auto-correlation is the matrix-valued function $\mathbf{\Sigma}_L(s, t) = \mathbb{E}\left[\mathbf{U}(\mathbf{X}_s, s)\mathbf{U}(\mathbf{X}_t, t)^T\right]$ where $\mathbf{X}_t$ satisfies the equation of the motion and the initial condition according to Eq.(1).

Remark 2.1. *Suppose $\mathbf{U}(\mathbf{x}, t)$ is a mean-zero, homogeneous, stationary, and divergence free random field on $\mathbb{R}^2$. Then the stochastic process $\{\mathbf{U}(\mathbf{X}_t, t), t \geq 0\}$ is a stationary process and so Lagrangian auto-covariance depends on the difference $t' - s'$, see [8]. Let $t = t' - s'$, then we write*

$$\mathbf{\Sigma}_L(s', t') = \mathbf{\Sigma}_L(0, t' - s') = \mathbf{\Sigma}_L(t). \tag{3}$$

Remark 2.2. *Because $\mathbf{X}_t = \int_0^t \mathbf{U}(\mathbf{X}_s, s)ds$, we have $\mathbb{E}\left[\mathbf{X}_t\mathbf{X}_t^T\right] = \int_0^t \int_0^t \mathbf{\Sigma}_L(s - s')ds\, ds'$. Thus, we can see how quickly the particle moves away from the origin by calculating the function $\mathbf{\Sigma}_L(t)$.*

Plugging Eq.(2) into Eq.(3) and assuming finite joint moments, we explicitly get the expression

$$\mathbf{\Sigma}_L(t) = \frac{1}{N} \sum_{p_1, p_2 = 1}^{N} \mathbb{E}\left[R_{p_1} R_{p_2} \mathbf{\Theta}_{p_1} \mathbf{\Theta}_{p_2}^T \sin(\Phi_{p_1}) \sin(\mathbf{W}_{p_2} \cdot \mathbf{X_t} + \Phi_{p_2})\right]. \tag{4}$$

In general, it is not possible to evaluate the expectations in Eq.(4) since the distribution of the stochastic process $\{\mathbf{X}_t, t \geq 0\}$ is unknown and presumably a function of all random variables in the model. However, we know that $\mathbf{X}_0 = \mathbf{0}$ with probability 1. This suggests that we can expand the Taylor series for the Lagrangian auto-covariance $\mathbf{\Sigma}_L(t)$ around $t = 0$, as in Eq.(5):

$$\mathbf{\Sigma}_L(t) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{dt^m} \left(\mathbf{\Sigma}_L(t)\right)\big]_{t=0} t^m. \tag{5}$$

3 Calculating Taylor coefficients

In order to evaluate Taylor coefficients for the Lagrangian auto-correlation function we need to calculate derivatives of Eq.(4). Using usual calculus rules for multi-dimensional variables we can

get a formula for the m^{th}-order derivative of $\mathbf{\Sigma}_L(t)$. After calculating a few derivatives by hand, the structure of each derivative becomes clear as we will explain. See [7] for more details.

Let T be the tree diagram represented in Figure 1. The m^{th} level of T has $m!$ nodes, each one denoted by some constant $K_\alpha^\beta = \mathbf{W}_\alpha \cdot \mathbf{W}_\beta^\perp$, $1 \leq \alpha, \beta \leq N$, and each branch i of T from the root to level m will have associated with it a number K_i which equals the product of the K_α^β values along the $m+1$ nodes in that branch. We will use the following abbreviations: $S_{p_1} = \sin(\Phi_{p_1})$, $C_{p_1} = \cos(\Phi_{p_1})$, $S_{p_j} = \sin(\mathbf{W}_{p_j} \cdot \mathbf{X_t} + \Phi_{p_j})$, and $C_{p_j} = \cos(\mathbf{W}_{p_j} \cdot \mathbf{X_t} + \Phi_{p_j})$, for all integers $j > 1$.

Theorem 3.1. *The m^{th}-order derivative of $\mathbf{\Sigma}_L(t)$ can be written in terms of the $m!$ branches of T of order m as*

$$\frac{d^m \mathbf{\Sigma}_L(t)}{dt^m} = \frac{1}{N^{m/2+1}} \sum_{p_1,\ldots,p_{m+2}=1}^{N} \sum_{i \in T} \mathbb{E}\left[R_{p_1} \cdots R_{p_{m+2}} \mathbf{\Theta}_{p_1} \mathbf{\Theta}_{p_2}^T K_i S_{p_1} D^i (S_{p_2} \cdots S_{p_{m+1}}) S_{p_{m+2}}\right], \tag{6}$$

where $i = (i_3, \ldots, i_{m+2})$ is a multi-index which ranges over the routes from the root of the tree to the leaves, K_i is the product along branch i, and D^i is a multi-index differentiation operator.

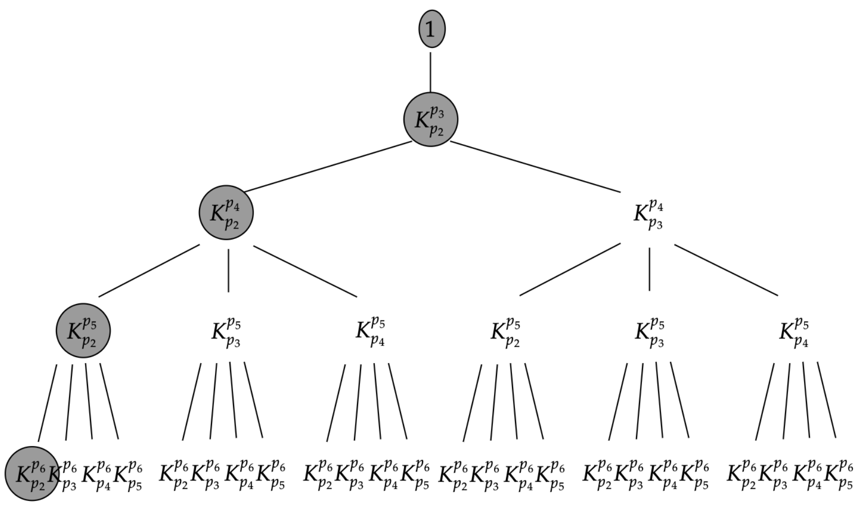

Figure 1: Tree diagram T with branches up to order 4.

Remark 3.1. *Notice that the total number of terms on the right side of Eq.(6) is $N^{m+2}m!$, where N is the number of Fourier modes and m is the derivative order. However, we will see that many terms are zero.*

Remark 3.2. *Figure 1 shows the tree T of order 4 with one branch highlighted. The product K_i for the highlighted branch of T is $K_i = 1 \cdot K_{\mathbf{p_2}}^{p_3} K_{\mathbf{p_2}}^{p_4} K_{\mathbf{p_2}}^{p_5} K_{\mathbf{p_2}}^{p_6}$. The multi-index for this branch is $i = (2,2,2,2)$. The product down the next branch of T is $K_i = 1 \cdot K_{\mathbf{p_2}}^{p_3} K_{\mathbf{p_2}}^{p_4} K_{\mathbf{p_2}}^{p_5} K_{\mathbf{p_3}}^{p_6}$ for $i = (2,2,2,3)$.*

Remark 3.3. *Notice that $K_\alpha^\alpha = \mathbf{W}_\alpha \cdot \mathbf{W}_\alpha^\perp = 0$. Thus, when summing over $p_1, \ldots, p_{m+2}$, it often happens that one of the factors in the product down a branch of T is 0, and so the whole term is 0.*

Remark 3.4. *The differential operator D^i is a product of derivatives of sine functions with respect to t given by*

$$D^i(S_{p_2} \ldots S_{p_{m+1}}) = D^{\eta_2}(S_{p_2}) \ldots D^{\eta_{m+1}}(S_{p_{m+1}}), \tag{7}$$

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where η_j is the total number of times that index j appears in i and $D^0(S_*) = S_*$, $D^1(S_*) = C_*$, $D^2(S_*) = -S_*$, $D^3(S_*) = -C_*$, $D^4(S_*) = S_*$, and so on. For example, if $i = (2, 2, 3, 4)$ then we have that $D^i(S_{p_2}S_{p_3}S_{p_4}S_{p_5}) = D^2(S_{p_2})D^1(S_{p_3})D^1(S_{p_4})D^0(S_{p_5}) = -S_{p_2}C_{p_3}C_{p_4}S_{p_5}$.

Using that collections of random variables $(R_1, \mathbf{W}_1, R_2, \mathbf{W}_2, \cdots, R_N, \mathbf{W}_N)$ and $(\Phi_1, \Phi_2, \ldots, \Phi_N)$ are independent and evaluating Eq.(6) at $t = 0$ gives

$$\frac{d^m \mathbf{\Sigma}_L(t)}{dt^m}\Big]_{t=0} = \frac{1}{N^{m/2+1}} \sum_{p\in\mathcal{P}} \sum_{i\in T} \mathbb{E}\left[\mathbf{G}_i^p(R, \mathbf{W})\right] \mathbb{E}\left[F_i^p(S, C)\right], \tag{8}$$

where $p = (p_1, p_2, \ldots, p_{m+2}) \in \mathcal{P}$ with $\mathcal{P} = \{1, 2, \ldots, N\}^{m+2}$, $\mathbf{G}_i^p(R, \mathbf{W}) = R_{p_1} \cdots R_{p_{m+2}} \mathbf{\Theta}_{p_1} \mathbf{\Theta}_{p_2}^T K_i$ is a function of random amplitudes and wave numbers, and $F_i^p(S, C) = S_{p_1} D^i(S_{p_2} \cdots S_{p_{m+1}}) S_{p_{m+2}}$ is a function of random phases, for each $i \in T$. Note that because we are evaluating at $t = 0$, the abbreviation S_{p_j} now equals $\sin(\Phi_{p_j})$ and C_{p_j} equals $\cos(\Phi_{p_j})$.

The general results above become much clearer when written out for specific values of m and i. Conversely, it is by doing specific calculations that the general result is found and understood. We also wrote code to generate symbolic expressions for each term. The hand calculations for small orders allow us to validate our code. The coded procedure is much faster than hand calculations and reduces the risk of manual computation error. Using the program, we get symbolic expressions for terms $\mathbb{E}\left[\mathbf{G}_i^p(R, \mathbf{W})\right] \mathbb{E}\left[F_i^p(S, C)\right]$, for $m = 2, \ldots, 10$.

Example 3.1. *Consider the case when $m = 4$, that is, to evaluate the 4^{th}-order derivative of $\mathbf{\Sigma}_L(t)$ at $t = 0$. We can get an expression for all terms $\mathbb{E}\left[\mathbf{G}_p^i(R, \mathbf{W})\right] \mathbb{E}\left[F_p^i(S, C)\right]$, for each $i \in T$, in Eq.(8). For example, for $i = (2, 2, 2, 2)$ we have*

$$\mathbb{E}\left[\mathbf{G}_p^i(R, \mathbf{W})\right] \mathbb{E}\left[F_p^i(S, C)\right] = \mathbb{E}\left[R_{p_1} \ldots R_{p_6} \mathbf{\Theta}_{p_1} \mathbf{\Theta}_{p_2}^T K_{p_2}^{p_3} K_{p_2}^{p_4} K_{p_2}^{p_5} K_{p_2}^{p_6}\right] \mathbb{E}\left[S_{p_1} S_{p_2} S_{p_3} S_{p_4} S_{p_5} S_{p_6}\right]. \tag{9}$$

Our code produces the output in Figure 2 for this term.

```
Output 01 of 24, for i=( 2 2 2 2):

+1 * E[Rp1 Rp2 Rp3 Rp4 Rp5 Rp6 * Thetap1 Thetap2^T * (Kp2^p3)(Kp2^p4)(Kp2^p5)(Kp2^p6)] * E[Sp1 Sp2 Sp3 Sp4 Sp5 Sp6]

Output 13 of 24, for i=( 2 3 2 2):

-1 * E[Rp1 Rp2 Rp3 Rp4 Rp5 Rp6 * Thetap1 Thetap2^T * (Kp2^p3)(Kp3^p4)(Kp2^p5)(Kp2^p6)] * E[Sp1 Cp2 Cp3 Sp4 Sp5 Sp6]
```

Figure 2: Computer output for $m = 4$ and $i = (2, 2, 2, 2)$.

In the same way, we can obtain an expression for all terms in the summation, for each $i \in T$ in Eq.(8). Notice that the 4^{th}-level of T has 24 branches and so the total number of terms like Eq.(9) is 24, as i varies over T, and for each value of i, there are many ways to choose the numbers $p_1, \ldots, p_6$ from 1 to N.

Recall that the random phases Φ_{p_j} are independent and uniformly distributed on $[0, 2\pi]$. So we can evaluate the trigonometric expectation factors $\mathbb{E}\left[F_i^p(S, C)\right]$ in Eq.(8) since we know their distributions. In fact, if $m \geq 1$ is an odd integer then there are an odd number of trigonometric factors to integrate over, which integrates to 0. Thus all factors $\mathbb{E}\left[F_i^p(S, C)\right]$ are 0 and so all odd-order Taylor coefficients in the expansion of the Lagrangian auto-covariance function $\mathbf{\Sigma}_L(t)$ are also 0, see [7]. Therefore, we can just focus on even-order Taylor coefficients. Many of the $\mathbb{E}\left[F_i^p(S, C)\right]$ factors will again be zero. Taking Eq.(9) as an example, if the numbers $p_1, p_2, \ldots, p_6$ are all distinct, the expected value will be zero. The numbers $p_1, \ldots, p_6$ need to be equal in pairs, or fours, or sixes to get non-zero integrals, and sine and cosine terms must also occur in pairs. Thus, many of the individual terms in Eq.(8) are zero.

3.1 Velocity fields with many Fourier modes

Looking further at terms involving trigonometric expectations such as $\mathbb{E}\left[F_i^p(S,C)\right]$, for $i \in T$, in Eq.(8), we realize that the number of terms with the p_j equal in fours or sixes are much smaller than the normalization factor N^{m+2}, see [7]. In fact, allowing the number of Fourier modes N to go to infinity, only trigonometric expectations having pairwise equal random phases as their arguments need to be taken into consideration.

For the rest of the paper we assume that the random variables $(R_1, \mathbf{W}_1), \ldots, (R_N, \mathbf{W}_N)$ are independent and identically distributed. This allows us to recognize that as we sum over $p_1, \ldots, p_{m+2}$ going from 1 to N, many of the terms have the same numerical values.

Remark 3.5. *Notice that we only need to consider indices $p_1, \ldots, p_{m+2}$ taking numerical values in the set $\{1, 2, \ldots, m/2+1\}$ since random variables are identically distributed. Moreover, there are $(m+1)(m-1)\cdots 3 \cdot 1$ distinct ways to choose such numerical values. Let $\mathcal{D}$ be the set of all distinct ways of arranging numbers $\{1, 2, \ldots, m/2+1\}$ in $m/2+1$ positions. For each $p \in \mathcal{D}$, there are $N(N-1)\ldots(N-m/2)$ terms as $\mathbb{E}\left[\mathbf{G}_i^p(R,\mathbf{W})\right]\mathbb{E}\left[F_i^p(S,C)\right]$ in Eq.(8), and all of them are equal numerically. After taking the limit as $N \to \infty$, we can write Eq.(8) as*

$$\lim_{N\to\infty} \frac{d^m \mathbf{\Sigma}_L(t)}{dt^m}\Big]_{t=0} = \sum_{i\in T}\sum_{p\in\mathcal{D}} \mathbb{E}\left[\mathbf{G}_p^i(R,\mathbf{W})\right]\mathbb{E}\left[F_p^i(S,C)\right], \tag{10}$$

and the right hand side of Eq.(10) has $(m+1)(m-1)\cdots 1 \cdot m!$ terms, which does not depend on N.

Example 3.2. *For $m = 4$, we can take $p = (p_1, p_2, \ldots, p_6) \in \mathcal{D}$, which is the set of all distinct ways of arranging numbers $\{1, 2, 3\}$ in 6 positions. In this case, the set $\mathcal{D}$ has $5 \cdot 3 \cdot 1 = 15$ elements:*

$$\begin{aligned}\mathcal{D} = \{&(1,2,3,3,2,1),(1,2,1,3,3,2),(1,2,3,2,1,3),(1,1,2,3,2,3),(1,2,2,1,3,3),\\ &(1,2,3,2,3,1),(1,2,3,1,3,2),(1,2,3,1,2,3),(1,2,1,3,2,3),(1,2,1,2,3,3),\\ &(1,2,2,3,3,1),(1,2,3,3,1,2),(1,1,2,3,3,2),(1,2,2,3,1,3),(1,1,2,2,3,3)\}.\end{aligned} \tag{11}$$

In Figure 3, we have symbolic expressions for terms as in Eq.(9) when p varies over the set $\mathcal{D}$. Notice that all but three terms in the list are 0 because some factor $K_\alpha^\alpha = 0$, for some $\alpha = 1, 2, 3$.

```
Output 01 of 24, for i=( 2 2 2 2):
General term:
+1 * E[Rp1 Rp2 Rp3 Rp4 Rp5 Rp6 * Thetap1 Thetap2^T * (Kp2^p3)(Kp2^p4)(Kp2^p5)(Kp2^p6)] * E[Sp1 Sp2 Sp3 Sp4 Sp5 Sp6]
Specific terms for 15 choices of p, with Theta_p^q as an abbreviation for Theta_p Theta_q^T:
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S3 S3 S2 S1] = 0
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S1 S3 S3 S2] = 0
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S3 S2 S1 S3] = 0
 1 * E[R1^2 R2^2 R3^2 * Theta_1^1 * (K_1^2)(K_1^2)(K_1^3)(K_1^3)] * E[S1 S1 S2 S3 S2 S3] = ?              (line 04)
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S2 S1 S3 S3] = 0
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S3 S2 S3 S1] = 0
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S3 S1 S3 S2] = 0
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S3 S1 S2 S3] = 0
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S1 S3 S2 S3] = 0
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S1 S2 S3 S3] = 0
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S2 S3 S3 S1] = 0
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S3 S3 S1 S2] = 0
 1 * E[R1^2 R2^2 R3^2 * Theta_1^1 * (K_1^2)(K_1^2)(K_1^3)(K_1^3)] * E[S1 S1 S2 S3 S3 S2] = ?              (line 13)
-1 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_2^2)(K_2^3)(K_2^3)] * E[S1 S2 S2 S3 S1 S3] = 0
 1 * E[R1^2 R2^2 R3^2 * Theta_1^1 * (K_1^2)(K_1^2)(K_1^3)(K_1^3)] * E[S1 S1 S2 S2 S3 S3] = ?              (line 15)
```

Figure 3: Computer output for $m = 4$, $i = (2, 2, 2, 2)$, ranging over the 15 choices of $p \in D$.

Remark 3.6. *Notice that in order to calculate the 4^{th}-order derivative of $\mathbf{\Sigma}_L(t)$ at $t = 0$ we need to evaluate 360 terms since T has 24 elements and $\mathcal{D}$ has 15 elements. However, many of these*

terms are 0. *In fact, after listing all non-zero terms we end up with a list having* 39 *terms. We also can relabel some indices and sort factors to group similar terms. For example, the terms in lines* 13 *and* 15 *in Figure 3 have the same numerical value. Finally, we end up with a list having only* 4 *distinct terms, as shown in Figure 4. The first term is non-zero, but because the same term occurs multiple times and* $K_\alpha^\beta = -K_\beta^\alpha$, *its coefficient happens to be zero. We refer to it as a non-null term. This is yet another way that terms drop out of the large sum we must calculate.*

```
Distinct terms for m=4:

 0 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^3)(K_2^3)(K_2^3)(K_2^3)] * E[S1^2]^3
 1 * E[R1^2 R2^2 R3^2 * Theta_1^1 * (K_1^2)(K_1^2)(K_2^3)(K_2^3)] * E[S1^2]^3
 3 * E[R1^2 R2^2 R3^2 * Theta_1^1 * (K_1^2)(K_1^2)(K_1^3)(K_1^3)] * E[S1^2]^3
-5 * E[R1^2 R2^2 R3^2 * Theta_1^2 * (K_1^2)(K_1^2)(K_1^3)(K_2^3)] * E[S1^2]^3
```

Figure 4: Computer output of distinct terms for $m = 4$.

Remark 3.7. *The symbolic expressions in Figure 4 depend on the joint distribution of the random variables* $(R_n, \mathbf{W}_n)$ *and the numerical value of* $\mathbb{E}\left[S_1^2\right]$, *which is 0.5. To calculate numerically, recognize that the Theta factors are 2 by 2 matrices made up of components of the wave number* $\mathbf{W}$, *and the K factors are dot products of wave numbers. One can multiply these out, use linearity to separate out terms, then use independence of the* $(R_n, \mathbf{W}_n)$ *to get a product of expected values. This reduces the computation to joint moments of R and* $\mathbf{W}$. *The highest moment encountered will be* $\mathbb{E}\left[R^2|\mathbf{W}|^{m+2}\right]$. *Therefore, the methodology we use to obtain Taylor coefficients for the Lagrangian auto-covariance function is quite general and allows us to explore a large variety of scenarios just by changing the joint distribution of random amplitudes and random wave numbers.*

3.2 Results

In Table 1, we present a summary for the number of terms that each derivative order demands to be numerically evaluated up to order $m = 10$. We notice the number of distinct non-null terms in each Taylor coefficient is much smaller than the total number of terms.

Table 1: Summary for non-zero terms.

Order	# terms	# non-zero terms	% non-zero terms	# distinct non-null terms
2	6.00×10^{00}	1.00×10^{0}	16.67%	1
4	3.60×10^{02}	3.90×10^{1}	10.83%	3
6	7.56×10^{04}	5.00×10^{3}	06.61%	22
8	3.81×10^{07}	1.43×10^{6}	03.75%	295
10	3.77×10^{10}	7.92×10^{8}	02.10%	6363

3.3 Some limitations

Notice that the number of terms needed to evaluate the m^{th}-order derivative of the Lagrangian auto-covariance function is $(m+1)(m-1)\cdots 1 \cdot m!$, according to Eq.(10). This demands a lot of computational work even for small order derivatives. We are able to perform such calculations up to order $m = 10$, which means we are able to numerically calculate only the first five non-zero terms of the Taylor expansion. So far, there is no shortcut to obtain just the non-zero terms. Moreover, it seems that the Taylor series converges slowly, when it does converge, so it is fundamental to be able to evaluate higher order Taylor coefficients.

4 Conclusion

In this work, we expand the Lagrangian auto-covariance function $\mathbf{\Sigma}_L(t)$ as a Taylor series for a random velocity field $\mathbf{U}(\mathbf{x}, t)$ written as a sum of N Fourier modes. To evaluate the m^{th} Taylor coefficient we start with an expression that depends on $N^{m+2}m!$ terms. Assuming some mild conditions on the parameters of the model and letting the number of Fourier modes go to infinity, then the number of terms needed to evaluate the m^{th}-order derivative of the Lagrangian auto-covariance is $(m+1)(m-1)\cdots 1 \cdot m!$, which is large but no longer depends on N. Using an appropriate programming language we are able to obtain all those terms, for small derivative orders, and write them down as symbolic expressions or as strings of characters. This allows us to identify terms that are equal to zero and to combine like terms, and thus drastically reduce the number of distinct terms. Moreover, we can convert these symbolic expressions into executable command lines to numerically calculate Taylor coefficients.

We are able to explore different scenarios, for different configurations of the parameters of the random velocity field, and get the Taylor expansion for the Lagrangian auto-covariance function just by setting new distributions for the parameters in the model. In this way, we can generate the Lagrangian auto-covariance function much faster than using Monte Carlo simulations for a large variety of parameter combinations.

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