An Approach to Partial Quadratic Eigenvalue Assignment of Vibration Systems Using Sylvester equations

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Abstract. This article proposes an approach to resolve the partial eigenvalue assignment output feedback control problem for second-order damped vibration systems. The matrices are prescribed in advance and highly depend on controllability conditions, and the system's observability eigenvalues are assigned. In addition, the real value spectral decomposition $P(\lambda)$ is exploited to establish conditions so that the feedback gain matrices do not spill over eigenvalue assignment. Thus, the numerical method presented applies to the active vibration control design of multiple inputs and outputs of functional engineering structures. However, it should be noted that the proposed algorithm can present complex computational problems if the mass matrix of the system is almost singular, as it involves the inverse calculation of the mass matrix. Two theorems were presented using Sylvester equations. Three algorithms were implemented based on the Sylvester equation, and examples were presented with their conclusions.

Keywords. Sylvester equations, Partial eigenvalue assignment, Output feedback control.

1 Introduction

Second-order linear systems capture the dynamic behavior of many natural phenomena have therefore found applications in vibration and structural analysis. Second-order linear systems have found applications in fields such as spacecraft control and robotic control. Therefore, they have

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attracted a lot of attention, in [1], [2], [3], [4], [5]. In [8] he presented an impulse elimination approach for partial assignment of eigenvalues to the descriptor system with the condition of $S-$controllability. In [7] presented an approach for assigning partial eigenvalues in second-order singular systems via proportional controller, derived control, and output feedback controller.

A solution of the generalized Sylvester equation is linked to a linear singular system subjected to any restriction on poles classification and regional placement. The problem of partially eigenvalue allocation for amortization vibrancy in the second-order system by static output feedback in [9]. In this note, it is considered that the controller follows a second-order singular linear system:

$$
\begin{aligned}
M_0\ddot{x} + D_0\dot{x} + N_0x &= Bu \\
y_0 &= C_0x \\
y_1 &= C_1\dot{x} \\
y_2 &= C_2\ddot{x}
\end{aligned}
\tag{1}
$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state vector and the control vector, respectively, and $M_0, D_0, N_0 \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$, $C_0, C_1, C_2 \in \mathbb{R}^{p \times n}$ are the system coefficient matrices. In particular demand, the matrices M_0, D_0, and N_0 being denominated the mass matrix, the structural damping matrix and the stiffness matrix, respectively. The coefficient matrix satisfies the follow assumption. A_1: rank(M) $= q$, $0 < q \leq n$, $rank(B) = m$, and $rank(C_0) = rank(C_1) = rank(C_2) = p$.

As for the control of the second-order linear system (1), more of the results are focus on stabilization), pole assignment [3], [4], and partial pole assignment, [5]. In addition, several theoretical results for second-order systems are progressed over the convenient enlarged first-order singular state-space model. The system (1), is expressed in the form:

$$
E_d\dot{z} = A_dz + B_du
\tag{2}
$$

with

$$
E_d = \begin{bmatrix} I_n & 0 \\ 0 & M_0 \end{bmatrix} \quad ; \quad A_d = \begin{bmatrix} 0 & I \\ -N_0 & -D_0 \end{bmatrix} \quad B_d = \begin{bmatrix} 0 \\ B \end{bmatrix}
\tag{3}
$$

$\dot{z} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$ and $z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

Therefore, these results eventually involve manipulations on $2n$ dimensional matrices E_d, A_d and B_d.

The remaining of the paper is organized in the following Sections: Section 2 in presented problem formulation. Section 3 is presented Eigenstructure by Sylvester equation . Section 4 is presented Numerical Algorithm of Partial Eigenvalue Assignment by SOF (Static Output Feedback). Section 5 is presented Numerical Examples. Finally, Section 6 concludes the paper.

2 Problem formulation

For the second-order descriptor dynamical system (1), by choosing the following control law:

$$
u(t) = -K_0y_0(t) - K_1y_1(t) - K_2y_2(t)
\tag{4}
$$

with $K_0, K_1, K_2 \in \mathbb{R}^{p \times n}$. It obtains the closed-loop system as follows:

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$$(M_0 + BK_2C_2)\ddot{x} + (D_0 + BK_1C_1)\dot{x} + (N_0 + BK_0C_0)x = 0 \tag{5}$$

System (5) can be written in the first-order state-space form

$$E_c\dot{z} = A_cz; \tag{6}$$

with

$$E_c = \begin{bmatrix} I_n & 0 \\ 0 & (M_0 + BK_2C_2) \end{bmatrix} \quad and$$

$$A_c = \begin{bmatrix} 0 & I \\ -(N_0 + BK_0C_0) & -(D_0 + BK_1C_1) \end{bmatrix}$$

System (1) gives rise to the quadratic eigenvalue problem of the open-loop vibration system with solving the eigenvalues λ_f and the associated eigenvectors $x_f \neq 0$, which satisfy

$$P(\lambda_f)x_f = 0 \ f = 1, 2, \cdots 2n \tag{7}$$

where

$$P(\lambda) = \lambda^2 M_0 + \lambda D_0 + N_0 \tag{8}$$

By definition in [9],

$$\mu_i^2(M_0 + BK_2C_2)y_i + \mu_i(D_0 + BK_1C_1)y_i +$$
$$(N_0 + BK_0C_0)y_i = 0, \ i = 1, 2, \cdots, p. \tag{9}$$

In general, the open-loop $2n$ eigenvalues are also called the open-loop poles of system (1). Correspondingly, the system (5) leads to the closed-loop quadratic eigenvalue problem.

$$P(\lambda)y = (\lambda^2(M_0 + BK_2C_2) + (D_0 + BK_1C_1)\lambda + (N_0 + BK_0C_0))y = 0 \tag{10}$$

3   Eigenstructure by Sylvester equation

The system (5) can be written in the first-order state-space form (6) and (7). Thus, for obtaining the output feedback matrix $F \ \sigma(E_p, A_p + B_pFC_p) \in \mathcal{C}^-$, is used the Sylvester equation in [6].

Consider the following linear time-invariant descriptor system in [6].

$$E_p\dot{x}(t) = A_px(t) + B_pu(t) \tag{11}$$
$$y(t) = C_px(t)$$

The Sylvester equations in [6].

$$A_pV_p - E_pV_pH_V = -B_pW_p, \quad \sigma(H_V) \in \mathcal{C}^-. \tag{12}$$
$$P_pA_p - H_PP_pE_p = -U_pC_d, \quad \sigma(H_P) \in \mathcal{C}^-. \tag{13}$$
$$P_pE_pV_p = 0 \tag{14}$$

The theorem 3.1 is based in[6], [9].

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Theorem 3.1. *Let (1), be S-controllable, and $V_p \in \mathbb{R}^{n \times p}$ and $W_p \in \mathbb{R}^{m \times p}$ satisfy the equation (12). Then, the following hold.*
1) The matrices V_p and W_p given by (12),

$$\begin{bmatrix} A_p - \lambda_i E_p & B_p \\ P_p E_p & 0 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0 \ i = 1, 2, \cdots, q. \tag{15}$$

satisfy Sylvester matrix equation (12) for, $i = 1, 2, \cdots, q$,
2) When

$$rank(\begin{bmatrix} V_i \\ W_i \end{bmatrix}) = m \ i = 1, 2, \cdots, q. \tag{16}$$

hold, (14) gives all the solutions.

Proof. Based in [6]. $\square$

4 Numerical Algorithm of Partial Eigenvalue Assignment by SOF

4.1 A sufficient condition and the associated matrix equation

For the acceleration, velocity and displacement output feedback, substituting $\Delta N = BK_0 C_0$; $\Delta D = BK_1 C_1$, $\Delta M = BK_2 C_2$ in [9] is obtained

$$\begin{bmatrix} K_0 & K_1 & K_2 \end{bmatrix} \begin{bmatrix} \overline{C}_0 \\ \overline{C}_1 \\ \overline{C}_2 \end{bmatrix} = 0 \tag{17}$$

The matrices K_0, K_1, K_2 are obtain from the homogeneous matrix equation (17) in [9]. The coefficient matrix of (17) in [9] involves only those few eigenvalues and the corresponding eigenvectors to be assigned and the analytical matrices of the system (1), and is a real matrix. Let $rank(A) = q$, then the matrix equation 17 in [9] has non-trivial solutions.

4.2 Numerical algorithm

We first present an approach to the general solutions of (17) in [9] for K_0, K_1, K_2. It is to determine the left null space of the coefficient matrix A using the singular value decomposition. Let Q denote the matrix that its rows are comprised of orthonormal basis vectors of the null space, and it is has

$$[K_0, K_1, K_2] = VQ \tag{18}$$

where the parametric matrix V is to be determined below. Secondly, K_0, K_1, K_2 must also implement some given eigenvalues assignment. The projection is characterized by the parametric vector μ_i where

$$\mu_i = (\Gamma^T(\lambda_i)(\lambda_i))^{-1}\Gamma^T(\lambda_i)x_i \tag{19}$$
$$y_i = \Gamma(\lambda_i)\mu_i \tag{20}$$
$$w_i = \Phi(\lambda_i)\mu_i \tag{21}$$

$$V(N[\overline{CY}]) \;=\; \overline{W} \qquad (22)$$

Equation (22) is of the form $ZE = S$, where S and E are given matrices of appropriate dimensions, and the matrix Z needs to be determined. The necessary and sufficient condition for the existence of solutions on this type of matrix equation is $SE^+E = S$ where the superscript $^+$ denotes the Moore-Penrose inverse of a matrix, and it is obtained the minimal S-norm solution

$$V \;=\; \overline{W}(N[\overline{CY}])^+ \qquad (23)$$

Substituting the obtained V back into (18), then output feedback gain matrices K_0, K_1, K_2 are eventually determined, in the closed-loop quadratic pencil $P(\lambda)$ of (8). When M_0 is nonsingular, this requirement guarantees that no impulses occur in the closed-loop system owing to infinite eigenvalues. It naturally involves with the problem of finding the distance of M_0 to the nearest singular matrix. The matrices K_0, K_1 and K_2 must also implement some given eigenvalues assignment. The theorem 4.1 is based in [6], [9].

Theorem 4.1. *Let (1), be S-controllable, and $V_p \in \mathbb{R}^{2n \times p}$ and $W_p \in \mathbb{R}^{m \times p}$ satisfy the equation (12). Then, the following hold.*
1) The matrices V_p and W_p given by (12),

$$\begin{bmatrix} A_p - \lambda_i E_p & B_p \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0 \; i = 1, 2, \cdots, q. \qquad (24)$$

satisfy Sylvester matrix equation (12) for, $i = 1, 2, \cdots, q$,
2) When

$$rank(\begin{bmatrix} V_i \\ W_i \end{bmatrix}) = m \; i = 1, 2, \cdots, q. \qquad (25)$$

3) K_2, K_1, K_0 is such that it satisfies

$$(M_0 + B_p K_2 C_2)\ddot{q}(t) + (D_0 + B_p K_1 C_1)\dot{q}(t) +$$
$$(N_0 + B_p K_0 C_0)q(t) = 0 \qquad (26)$$

Proof. Based in [6], [7], [8], [9]. □

    Algorithm basic $Z1$ Input: $M_0, D_0, N_0, B_p, C_0, C_1, C_2$ Output: K_0, K_1, K_2
    Step (1) Compute S_1 and the left null space Q of the coefficient matrix in Equation (18).
    Step (2) Compute $y_i, w_i, i = 1, 2, \cdots, p$ by Equations (24), (25), (20) to form Y and W.
    Step(3) Solve Equation (22) for V with Equation (23) after checking the existence of the solution.
    Step (4) Substitute V back into Equation (18) to give K_0, K_1, K_2.

5 Numerical Examples

To demonstrate the performance of the present approach, two numerical examples are analyzed in this section. All codes are run in MATLAB with machine precision 10^{-16} on a personal computer.

Consider a simple linear dynamical system (1) with $M_0, D_0, N_0, B, C_0, C_1, C_2$ in [9].

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Algorithm basic $Z1$ Input: $M_0, D_0, N_0, B, C_0, C_1, C_2$ Output: K_0, K_1, K_2

Step (1) Compute S_1 and the left null space Q of the coefficient matrix in Equation (18).

Step (2) Compute $y_i, w_i, i = 1, 2, \cdots, p$ by Equations (24), (25), (20) to form Y and W.

Step(3) Solve Equation (22) for V with Equation (23) after checking the existence of the solution.

Step (4) Substitute V back into Equation (18) with (21) to give K_0, K_1, K_2 and the matrix K_0, K_1, K_2 such that $K_2C_2V_2 = W_2$, $K_1C_1V_1 = W_1$, and $K_0C_0V_0 = W_0$,

$$V_0 = \begin{bmatrix} 0.0521304 & 0.0439039 & 0.0383248 & 0.0348497 \\ -0.076474 & -0.0719608 & -0.0708508 & -0.0726528 \\ 0.0070045 & 0.0070247 & 0.0066577 & 0.0061644 \\ 0.0043382 & 0.0040844 & 0.0037361 & 0.0028785 \\ 0.0017522 & 0.0004953 & -0.0002459 & -0.0006246 \\ 0.0074559 & 0.008552 & 0.0083765 & 0.0071757 \end{bmatrix}$$

$$W_0 = \begin{bmatrix} -0.3242153 & -0.2210717 & -0.2005616 & -0.273146 \\ -1.1272841 & -1.1120762 & -1.1044034 & -1.092142 \\ 0.3896079 & 0.3995321 & 0.4373715 & 0.4820908 \\ 0.6208033 & 0.6569759 & 0.6137823 & 0.5062952 \end{bmatrix}$$

$$K_0 = \begin{bmatrix} -6.1606949 & 5.4435459 & -14.456099 & 53.214836 \\ 4.1704143 & 13.62053 & -4.0775774 & -31.944519 \\ -9.9164149 & -12.267784 & -18.247316 & 33.716473 \\ 3.875054 & 3.1614239 & 31.649585 & 12.754786 \end{bmatrix}$$

where the eigenvalues are $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3, \lambda_4 = -4, \lambda_5 = -5, \lambda_6 = -6, \lambda_7 = -37.149593, \lambda_8 = -4.649849, \lambda_9 = -0.9921309 + 3.8887332j$ $\lambda_{10} = -0.9921309 - 3.8887332j$ $\lambda_{11} = -0.4872684 + 0.2499814j, \lambda_{12} = -0.4872684 - 0.2499814j.$

$$V_1 = \begin{bmatrix} 0.0446926 & 0.0374525 & -0.0291067 & -0.0236673 \\ -0.0783949 & -0.0746852 & 0.0726337 & 0.0678909 \\ 0.0038137 & 0.0046988 & -0.0012483 & -0.0011304 \\ 0.0060943 & 0.0051477 & -0.00533 & -0.0026561 \\ 0.0018332 & 0.000595 & -0.0000931 & 0.0001565 \\ 0.0069051 & 0.0089582 & 0.0038211 & 0.0050015 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} 0.5059455 & -0.4979435 & 0.7124262 & 0.8776696 \\ -1.1048955 & -1.0545573 & 0.8970177 & 0.7237905 \\ 0.3395982 & 0.2477809 & -0.6899967 & -0.6205126 \\ 0.5704727 & 0.6765353 & 0.2266264 & 0.3581733 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 5.9149083 & 19.2191996 & 3.5612338 & 1.056327 \\ -1.8569263 & 7.5686617 & -39.087756 & -45.860668 \\ -11.322169 & -13.724264 & -107.379012 & 9.407447 \\ 17.7752241 & 6.5700072 & 30.3076732 & 279694 \end{bmatrix}$$ where the eigenvalues are $\lambda_1 = -1, \lambda_2 = -2,$

$\lambda_3 = -3, \lambda_4 = -4, \lambda_5 = -5, \lambda_6 = -6, \lambda_7 = -38.653889, \lambda_8 = -3.695664, \lambda_9 = -2.8420891 + .6086082j$ $\lambda_{10} = -2.8420891 - .6086082j.$

$$V_2 = \begin{bmatrix} 0.0488385 & 0.0403992 & -0.025387 & -0.0181925 \\ -0.0797826 & -0.0784526 & 0.0687017 & 0.0611566 \\ 0.0038728 & 0.0040052 & -0.0006003 & -0.0004777 \\ 0.0056806 & 0.005733 & -0.0052843 & -0.0023258 \\ 0.0017419 & 0.000654 & -0.0001266 & 0.0000899 \\ 0.0066251 & 0.0062879 & 0.0053903 & 0.0062784 \end{bmatrix} ;$$

$$W_2 = \begin{bmatrix} -0.440617 & -0.4841541 & 0.777753 & 0.9964393 \\ -1.1356319 & -1.1110555 & 0.8038804 & 0.5579771 \\ 0.3574743 & 0.3837874 & -0.7035365 & -0.5846476 \\ 0.5476882 & 0.4999696 & 0.3360751 & 0.449065 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 7.7654659 & 19.487249 & 1.609387 & 96.48342 \\ -3.4471017 & 5.1972494 & -58.282258 & -59.589812 \\ -4.6306656 & -8.6625354 & -57.413782 & 35.723326 \\ 9.2594212 & 10.475041 & 163.1129 & 29.705745 \end{bmatrix}$$ where the eigenvalues are

$\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3, \lambda_4 = -4, \lambda_5 = -5, \lambda_6 = -6,$

$\lambda_1 = -3, \lambda_2 = -4, \lambda_3 = -22.229984, \lambda_4 = -0.509633, \lambda_5 = -1.5034249, \lambda_6 = -5.2611072, \lambda_7 = -4.1601947, \lambda_8 = -40.283544, \lambda_9 = -1.2514216 + 5.4279253j.$ $\lambda_{10} = -1.2514216 - 5.4279253j.$

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6   Conclusions

This article proposes an approach to resolve the partial eigenvalue assignment output feedback control problem for second-order damped vibration systems. So the matrices are simply prescribed in advance and highly dependent on the conditions of controllability and observability of the system where the eigenvalues are assigned. In addition, the real value spectral decomposition $P(\lambda)$ is exploited to establish conditions so that the feedback gain matrices do not spill over eigenvalue assignment. The numerical method presented applies to the active vibration control design of multiple inputs and outputs of practical engineering structures.

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