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Eigenvalue Assignment in Second order systems Using Sylvester equations: Approach

Elmer R. L. Villarreal [1]
Departamento de Ciências Naturais, Matemática e Estatística,
Universidade Federal Rural do Semi-árido DCME/UFERSA, Mossoró, RN.
Andres O. Salazar [2]
Departamento de Engenharia de Computação e Automação,
Universidade Federal do Rio Grande do Norte DCA/UFRN, Natal, RN.
Thiago F. do Nascimento [3]
Departamento de Engenharia de Computação e Automação,
Universidade Federal do Rio Grande do Norte DCA/UFRN, Natal, RN.
Lucas D. da Silva [4]
Engenharia Mecânica Centro de Engenharias ,
Universidade Federal Rural do Semi-árido EM/UFERSA, Mossoró, RN.
Walter M. Rodrigues [5]
Departamento de Ciências Naturais, Matemática e Estatística,
Universidade Federal Rural do Semi-árido, DCME/UFERSA, Mossoró, RN.

Abstract. Second-order systems are those whose models can be written by a second-order differential equation. That is, they are those that have two poles. The present article proposes an approach to solve the output feedback control with eigenvalues for second-order systems using Sylvester equations. The matrices are prescribed in advance and depend greatly on the controllability conditions, being assigned the system's observability eigenvalues. Furthermore, the real-value spectral decomposition $T(\lambda)$ is explored to establish conditions so that the feedback gain matrices do not overflow over the eigenvalue assignment. However, it should be noted that the proposed algorithms may present complex computational problems. Two theorems were presented using Sylvester equations. The algorithms were implemented based on Sylvester equations, and examples were presented with their conclusions.

Keywords. Sylvester equations, Eigenvalue assignment, Output feedback control, Second order system.

1 Introduction

Second-order linear systems capture the dynamic behavior of many natural phenomena, and have found applications in many fields, such as vibration and structural analysis, spacecraft control and robotics control and, hence, have attracted much attention [3]. In [6] presented impulse elimination approach to partial eigenvalue assignment for the descriptor system with the condition of

[1]elmerllanos@ufersa.edu.br
[2]andres@dca.ufrn.br
[3]thiago.figueiredo@ufrn.br
[4]lucasdantas742@gmail.com
[5]walterm@ufersa.edu.br

$S-$controllability. In [5] presented an approach for partial eigenvalue assignment in singular second order systems via proportional controller plus derived control plus output feedback controller.

The problem of partially eigenvalue allocation for amortization vibrancy in the second-order system by static output feedback in [8]. In this note, it is considered that the controller follows a second-order singular linear system:

$$\begin{aligned} P_0\ddot{x} + Q_0\dot{x} + R_0 x &= Bu \\ y_0 &= C_0 x \\ y_1 &= C_1\dot{x} \\ y_2 &= C_2\ddot{x} \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state vector and the control vector, respectively, and $P_0, Q_0, R_0 \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$, $C_0, C_1, C_2 \in \mathbb{R}^{p\times n}$ are the system coefficient matrices. In particular demand, the matrices P_0, Q_0, and R_0 being denominated the mass matrix, the structural damping matrix and the stiffness matrix, respectively. As for the control of the second-order linear system (1), more of the results are focus on stabilization), pole assignment [1], [2], and partial pole assignment, [3]. The system (1), is expressed in the form:

$$E_f\dot{z} = A_f z + B_f u \tag{2}$$

with

$$E_f = \begin{bmatrix} I_n & 0 \\ 0 & P_0 \end{bmatrix} \; ; \; A_f = \begin{bmatrix} 0 & I \\ -R_0 & -Q_0 \end{bmatrix} \; B_f = \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{3}$$

$\dot{z} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$ and $z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

Therefore, these results eventually involve manipulations on $2n$ dimensional matrices E_f, A_f and B_f.

The remaining of the paper is organized in the following Sections: Section 2 in presented problem formulation. Section 3 is presented Eigenstructure by Sylvester equation. Section 4 is presented Numerical Algorithm of Eigenvalue Assignment by SOF (Static Output Feedback) and numerical Examples. Finally, Section 5 concludes the paper.

2 Problem formulation

For the second-order descriptor dynamical system (1), by choosing the following control law:

$$u(t) = -L_0 y_0(t) - L_1 y_1(t) - L_2 y_2(t) \tag{4}$$

with $L_0, L_1, L_2 \in \mathbb{R}^{p\times n}$. It obtains the closed-loop system as follows:

$$(P_0 + BL_2C_2)\ddot{x} + (Q_0 + BL_1C_1)\dot{x} + (R_0 + BL_0C_0)x = 0 \tag{5}$$

System (5) can be written in the first-order state-space form

$$E_b\dot{z} = A_b z; \tag{6}$$

with

$$\begin{aligned} E_b &= \begin{bmatrix} I_n & 0 \\ 0 & (P_0 + BL_2C_2) \end{bmatrix} \ \text{and} \\ A_b &= \begin{bmatrix} 0 & I \\ -(R_0 + BL_0C_0) & -(Q_0 + BL_1C_1) \end{bmatrix} \end{aligned}$$

System (1) gives rise to the quadratic eigenvalue problem of the open-loop vibration system with solving the eigenvalues λ_k and the associated eigenvectors $x_k \neq 0$, which satisfy

$$T(\lambda_k)x_k = 0 \ k = 1, 2, \cdots 2n \tag{7}$$

where

$$T(\lambda) = \lambda^2 P_0 + \lambda Q_0 + R_0 \tag{8}$$

By definition in [8],

$$\begin{aligned} &\mu_i^2(P_0 + BL_2C_2)y_i + \mu_i(Q_0 + BL_1C_1)y_i + \\ &\quad (R_0 + BL_0C_0)y_i = 0, \ i = 1, 2, \cdots, p. \end{aligned} \tag{9}$$

In general, the open-loop 2n eigenvalues are also called the open-loop poles of system (1). Correspondingly, the system (5) leads to the closed-loop quadratic eigenvalue problem.

$$T(\lambda)y = (\lambda^2(P_0 + BL_2C_2) + (Q_0 + BL_1C_1)\lambda + (R_0 + BL_0C_0))y = 0 \tag{10}$$

3 Eigenstructure by Sylvester equation

The system (5) can be written in the first-order state-space form (6) and (7). Thus, for obtaining the output feedback matrix K $\sigma(E_s, A_s + B_sKC_s) \in \mathcal{C}^-$, is used the Sylvester equation in [4].

Consider the following linear time-invariant descriptor system in [4].

$$\begin{aligned} E_s\dot{x}(t) &= A_sx(t) + B_su(t) \\ y(t) &= C_sx(t) \end{aligned} \tag{11}$$

The Sylvester equations in [4].

$$A_sV_s - E_sV_sH_V = -B_sW_s, \qquad \sigma(H_V) \in \mathcal{C}^- \tag{12}$$

$$T_sA_s - H_TT_sE_s = -U_sC_s, \qquad \sigma(H_T) \in \mathcal{C}^- \tag{13}$$

$$T_sE_sV_s = 0 \tag{14}$$

The theorem 3.1 is based in [4], [8].

Theorem 3.1. *Let (1), be S-controllable, and $V_s \in \mathbb{R}^{n\times p}$ and $W_s \in \mathbb{R}^{m\times p}$ satisfy the equation (12). Then, the following hold.*

1) The matrices V_s and W_s given by (12),

$$\begin{bmatrix} A_s - \lambda_iE_s & B_s \\ T_sE_s & 0 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0 \ i = 1, 2, \cdots, q. \tag{15}$$

satisfy Sylvester matrix equation (12) for, $i = 1, 2, \cdots, q$,

2) When

$$rank(\begin{bmatrix} V_i \\ W_i \end{bmatrix}) = m \ i = 1, 2, \cdots, q. \tag{16}$$

hold, (14) gives all the solutions.

Proof. Based in [4]. □

Theorem 3.2. *Let* $E_s, A_s \in \mathbb{R}^{n\times n}$, $B_s \in \mathbb{R}^{n\times m}$, $H_V \in \mathbb{R}^{p\times p}$ *satisfy*

$$rank[A_s - \lambda E_s \ B_s] = n, \quad for \ any \ \lambda \in \sigma(H_V). \tag{17}$$

Further, let $H \in \mathbb{R}^{(n+m)\times m}[\lambda]$ *be a polynomial matrix satisfying* $[A_s - \lambda E_s \ B_s]H(\lambda) = 0_{n\times m}$.

Then:

(1) The matrices $V_s \in \mathbb{R}^{n\times p}$ *and* $W_s \in \mathbb{R}^{m\times p}$ *given by*

$$\begin{bmatrix} V_s \\ W_s \end{bmatrix} = Syl(H(\lambda), H_V, Z) \tag{18}$$

satisfy the matrix equation (12) for any matrix $Z \in \mathbb{R}^{m\times p}$. *(2) When* $rank H(\lambda) = r$ *for any* $\lambda \in \sigma(H_V)$, *all the matrices* V_s *and* W_s *satisfying the matrix equation (12) can be explicitly expressed by (18).*

Proof. Based in [4] and [7]. □

4 Numerical Algorithm

Let $G(\lambda)$ and X be defined as

$G(\lambda) = [A_s - \lambda E_s \ B_s]$; $X = \begin{bmatrix} V_s \\ W_s \end{bmatrix}$

The following basic procedure is proposed to calculate the feedback controller that stabilizes the closed loop system, when $m + p > q$. Closed loop eigenvalues are positioned arbitrarily close to the set ; they are symmetric sets of pre-specified eigenvalues. The (E_s, A_s, B_s, C_s) system is considered to be strongly controllable and strongly detectable.

Algorithm

Step 1: Choose an array $H_T \in \mathbb{R}^{q-p\times q-p}$ such that $\sigma(H_T) = \Lambda_T \in \mathcal{C}^-$ and Sylvester equation (13) is solved to find a matrix $T_s \in \mathbb{R}^{q-p\times n}$ such that

$$rank \left(\begin{bmatrix} T_s E_s \\ C_s \end{bmatrix} \right) = q \tag{19}$$

Step 2: Sylvester equation (12) is solved, for some $H_V \in \mathbb{R}^{p\times p}$ matrix such that $\sigma(H_V) = \Lambda_V \in \mathcal{C}^-$ taking into account that the matrix V_s must check the condition of the coupling (14) and taking into account that $rank\,(E_s V_s) = p$ .The condition (19) guarantees, in particular, that $rank(T_s E_s) = q$.

Step 3: By construction, the matrix V_s must verify that rank $(C_s V_s) = p$ and the matrix K can be calculated by:

$$K \ = \ W_s(C_s V_s)^{-1} \tag{20}$$

◇

Remark 4.1. *steps* 1 *and* 2 *can be solved using standard techniques for positioning the self-structure. Similar to the previous case, the matrices* V_s *and* W_s *used for the calculation of* K *can be constructed only with real elements. In particular: if* $\lambda_i \in \mathcal{C}^-$*, it is considered* $\lambda_{i+1} = \lambda_i^*$ *e* $\begin{cases} V_i = Re\,(v_i), & V_{i+1} = Imag\,(v_i) \\ W_i = Re\,(w_i), & W_{i+1} = Imag\,(w_i) \end{cases}$ *, where* V_i *and* W_i *denote the columns of the matrices* V_s *and* W_s*, respectively.*

In step 1, under the condition that the system is strongly observable (detectable). As will be seen later, degrees of freedom existing in the choice of V_s that satisfy the coupling condition $T_sE_sV_s = 0$, can also be used to guarantee obtaining K such that $KC_sV_s = W_s$ in [4].

4.1 Example

Consider a simple linear dynamical system (1) in [8]

$$P_0 = \begin{bmatrix} 1.56 & 0.66 & 0.54 & -0.39 & 0 & 0 \\ 0.66 & 0.36 & 0.39 & -0.27 & 0 & 0 \\ 0.54 & 0.39 & 3.12 & 0 & 0.54 & -0.39 \\ -0.39 & -0.27 & 0 & 0.72 & 0.39 & -0.27 \\ 0 & 0 & 0.54 & 0.39 & 3.12 & 0 \\ 0 & 0 & -0.39 & -0.27 & 0 & 0.72 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 0.72 & 1.08 & -0.72 & 1.08 & 0 & 0 \\ 1.08 & 2.16 & -1.08 & 1.08 & 0 & 0 \\ -0.72 & -1.08 & 1.44 & 0 & -0.72 & 1.08 \\ 1.08 & 1.08 & 0 & 4.32 & -1.08 & 1.08 \\ 0 & 0 & -0.72 & -1.08 & 1.44 & 0 \\ 0 & 0 & 1.08 & 1.08 & 0 & 4.32 \end{bmatrix}$$

$$R_0 = \begin{bmatrix} 12 & 18 & -12 & 18 & 0 & 0 \\ 18 & 36 & -18 & 18 & 0 & 0 \\ -12 & -18 & 24 & 0 & -12 & 18 \\ 18 & 18 & 0 & 72 & -18 & 18 \\ 0 & 0 & -12 & -18 & 24 & 0 \\ 0 & 0 & 18 & 18 & 0 & 72 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix};$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}; C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1. \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Considered the system in the equations (8), (9)

$$E_f = \begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix}; A_s = \begin{bmatrix} 0 & I_n \\ -R & -Q \end{bmatrix};$$

$$B_f = \begin{bmatrix} 0 \\ B \end{bmatrix} \; C_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T$$

Algorithm basic S1

Resolved the equation (11) for calculate the matrices W_s, V_s, satisfies the equation (14) and the matrix K, such that $KC_sV_s = W_s$:

$$T_s = \begin{bmatrix} 0.0191586 & 0.0067923 \\ 0.002596 & -0.0048225 \\ -0.1142892 & -0.0984266 \\ 0.0291704 & 0.0464583 \\ -0.1560708 & -0.1242605 \\ 0.1411607 & 0.1285409 \\ -0.041492 & -0.0123643 \\ 0.0355007 & 0.0117136 \\ 0.050413 & 0.0245556 \\ 0.0223914 & 0.0129728 \\ 0.0354971 & 0.0116523 \\ 0.0102292 & 0.0214763 \end{bmatrix}^T$$

$$V_s = \begin{bmatrix} 0.0521304 & 0.0439039 & 0.0383248 & 0.0348497 \\ -0.076474 & -0.0719608 & -0.0708508 & -0.0726528 \\ 0.0070045 & 0.0070247 & 0.0066577 & 0.0061644 \\ 0.0043382 & 0.0040844 & 0.0037361 & 0.0028785 \\ 0.0017522 & 0.0004953 & -0.0002459 & -0.0006246 \\ 0.0074559 & 0.008552 & 0.0083765 & 0.0071757 \\ -0.1563912 & -0.1756154 & -0.1916239 & -0.2090984 \\ 0.229422 & 0.2878434 & 0.3542541 & 0.435917 \\ -0.0210135 & -0.0280988 & -0.0332887 & -0.0369862 \\ -0.0130146 & -0.0163374 & -0.0186804 & -0.0172712 \\ -0.0052567 & -0.0019812 & 0.0012293 & 0.0037475 \\ -0.0223678 & -0.0342079 & -0.0418824 & -0.043054 \end{bmatrix}$$

$$W_s = \begin{bmatrix} -0.3242153 & -0.2210717 & -0.2005616 & -0.273146 \\ -1.1272841 & -1.1120762 & -1.1044034 & -1.092142 \\ 0.3896079 & 0.3995321 & 0.4373715 & 0.4820908 \\ 0.6208033 & 0.6569759 & 0.6137823 & 0.5062952 \end{bmatrix}$$

$$K = \begin{bmatrix} -6.1606949 & 5.4435459 & -14.456099 & 53.214836 \\ 4.1704143 & 13.62053 & -4.0775774 & -31.944519 \\ -9.9164149 & -12.267784 & -18.247316 & 33.716473 \\ 3.875054 & 3.1614239 & 31.649585 & 12.754786 \end{bmatrix}$$ where the eigenvalues are

$\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$, $\lambda_4 = -4$, $\lambda_5 = -5$, $\lambda_6 = -6$, $\lambda_7 = -4.649849$, $\lambda_8 = -37.149593$, $\lambda_9 = -0.9921309 + 3.8887332i$, $\lambda_{10} = -0.9921309 - 3.8887332i$, $\lambda_{11} = -0.4872684 + 0.2499814i$,

$\lambda_{12} = -0.4872684 - 0.2499814i.$

5 Conclusions

This article proposes an approach to resolve the eigenvalue assignment output feedback control problem for second-order systems using the Sylvester equations. So the matrices are simply prescribed in advance and highly dependent on the conditions of controllability and observability of the system where the eigenvalues are assigned. The numerical method presented applies to the active vibration control design of multiple inputs and outputs of practical engineering structures.

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