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On geometric invariants of plane curves

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Abstract. In this paper, we study some geometric invariants of closed plane curves, that can help us classify these curves. We focus on two invariants: the number of inflection points and the number of vertex points. We intend to find models of curves with a number of predefined double points and with the smallest possible number of inflection points and vertex points.

Palavras-chave. Geometric modeling, plane curves, inflections, vertices, graphs of stable maps.

1 Introduction

Stable applications from surfaces to the plane have been extensively studied (see for example [1, 5, 8, 11–13]). The image of the singular set of a stable map, from a closed surface M to the plane, known as the *apparent contour*, consists of a collection of closed curves, immersed in the plane, with possibly a finite number of transverse intersections and singularities. Invariants of the apparent contour carry substantial information about the surface M. The three well-known invariants in the study of apparent contour are the number of components of the singular set, the number of cusps and the number of double points. In [2], Arnold introduced Vassiliev-type invariants for the isotopic classification of stable closed curves and presented a table with possible closed, stable curves, immersed in the plane, with a maximum of five double points. In [7], the authors demonstrated that only six curves in the Arnold table in [2] can be the apparent contour of fold maps (maps with zero cusps) from sphere to the plane. It is notable that, In [2], Arnold presented a family of plane curves, called basic curves, with the smallest number of double points, within the class of immersions given by Whitney ([6]). These curves are labelled by K_i for $i \geq 0$ and are illustrated in Figure 1. Besides the topological invariants for closed plane curves, one

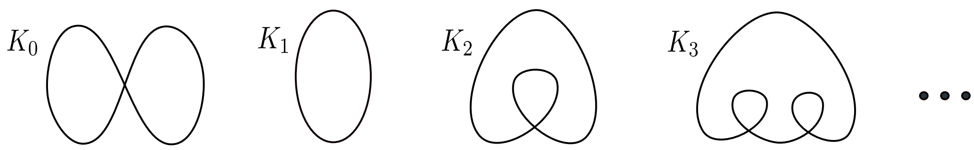

Figura 1: Arnold basic curves.

can consider geometric concepts of plane curves as well. More precisely, we concern the minimum

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number of inflection and vertex points of a closed stable plane curve. Recall that, a point p on a plane curve γ is called an ordinary inflection if the curvature of γ at p is zero. Also a point $q \in \gamma$ is called an ordinary vertex if the curvature of γ at q is not zero but the first derivative of curvature at q is zero. There are several works investigating the minimum number of inflection points of a Jordan curve. For a Jordan curve (Arnold curve of type K_1) there is a well known theorem called "the 4 vertex theorem". For an Arnold curve of type K_2, the minimum number of vertices is 2 (consider the curve $r = 1 - 2\sin(\theta)$ in polar coordinates). To the best of our knowledge, there is no sensible study on the minimum number of inflection and vertex points of a closed stable plane curve simultaneously. Thereupon, it is reasonable to ask whether, for a given closed plane curve γ, it is possible to determine a model curve with *minimum number of vertices (denoted by ν_γ) and inflections (denoted by ι_γ)*, which is isotopic to the curve or not. Also, is it possible to determine the set of plane curves considered as the apparent contour of some stable applications from closed surfaces to the plane, with the smallest number of vertex and inflection points?

In [10], the authors studied the local deformations of a family of plane curves, which considers the geometry of the plane curves (inflections and vertices) together with their singularities. In particular, they considered in [10] the codimension 1 transitions of a second order vertex, second order inflection and an ordinary cusp as illustrated in Figure 2. This motivated us to use the

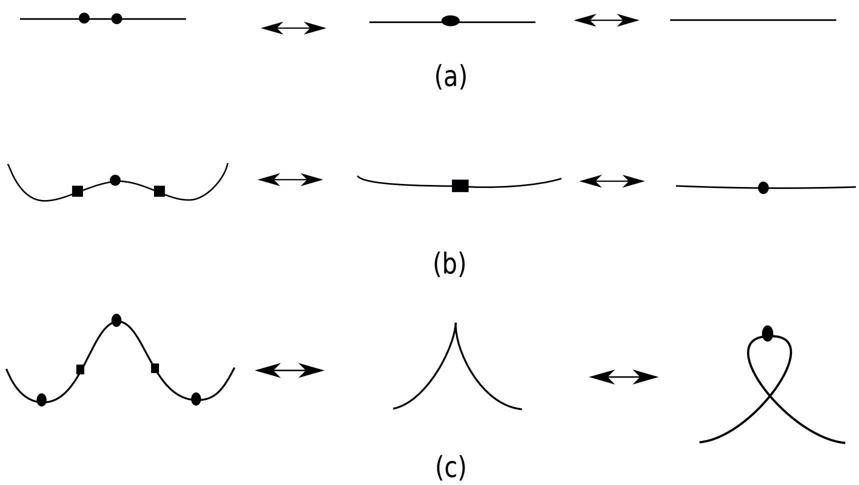


Figura 2: Codimension 1 geometric transitions: (a) second order vertex (b) second order inflection (c) ordinary cusp. Dots and squares represent ordinary vertex and inflection points respectively.

results given in [10] to answer the questions mentioned above.

Remark 1.1. *It is important to pay attention to the fact that the transitions given in Figure 2 are local in nature while the invariants ν_γ and ι_γ are global invariants of the plane curve γ. To solve this problem, in [9] we use some arguments similar to one given in [4] for converse of the four vertex theorem.*

Example 1.1. In [3, page 21], P. J. Giblin and J. W. Bruce presented a family of plane curves (called limaçon) given by

$$\gamma_a(t) = (a\cos(t) + \cos(2t) + 1, a\sin(t) + \sin(2t)) .$$

By changes of the variable a, they calculated the inflection and vertex points (see Figure 3). These deformations shows how a curve of type K_1 changes to a curve of type K_2.

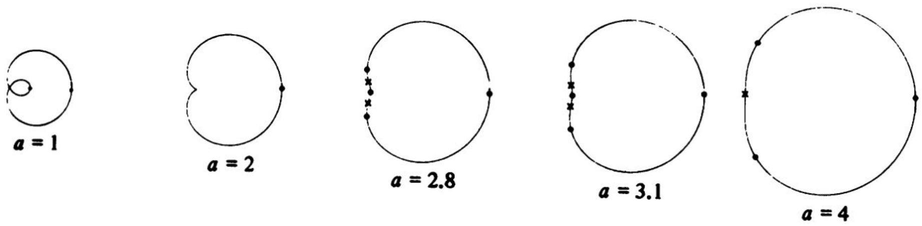
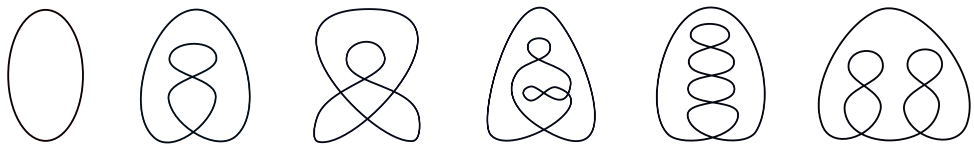


Figura 3: Deformations of a family of limaçons. Here dots represent vertices and cross-marks represent inflections. This Figure is given in [3].

As already mentioned above, the closed plane curves also can be seen as the apparent contour of stable maps from a closed oriented surface M to the plane $\mathbb{R}^2$. In [2], Arnold presented a table with the possible closed, stable curves, immersed in the plane, with a maximum of five double points. In [7], the authors proved that only six curves in the Arnold table in [2] may be the apparent contour of fold maps (a stable map without cusp points is called fold map) from the sphere to the plane. These are the curve K_1 (ellipse) and some other curves denoted by D_{i_j} given in Figure 4. In following theorems we present the invariants ν and ι for Arnold basic curves and the six curves



K_1 D_2 D_{4_1} D_{4_2} D_{4_3} D_{4_4}

Figura 4: Examples of plane curves which can be apparent contours of maps from the sphere to the plane $\mathbb{R}^2$.

given in Figure 4.

Theorem 1.1. *For the Arnold basic curves, K_i, we have:*

(a) $\iota(K_0) = 4$ and $\iota(K_i) = 0$, for $i \geq 1$;

(b) $\nu(K_1) = 4$, $\nu(K_0) = 2$ and $0 < \nu(K_i) \leq 2(i-1)$, for $i \geq 2$.

Theorem 1.2. *For every plane curve γ given in Figure 4, the geometrical invariants ν_γ and ι_γ are given in Table 1. In this table D denotes the number of self intersections.*

Tabela 1: Geometric invariants of apparent counter curves given in Figure 4.

Curve	D	ι	ν
K_1	0	0	4
D_2	2	2	2
D_{4_1}	4	2	2
D_{4_2}	4	4	4
D_{4_3}	4	6	6
D_{4_4}	4	4	4

Remark 1.2. *We may conjecture that in item (b) of Theorem 1.1, $\nu(K_2) = 2$ and consequently $\nu(K_i) = 2(i-1)$ for every $i \geq 2$.*

Example 1.2. *The idea of using the codimension 1 geometric transitions given in Figure 2 can be used to estimate the invariants ν and ι of any plane curve. Figure 5 illustrates an example of using a D_{4_1} curve to estimate the geometric invariants of the plane curve.*

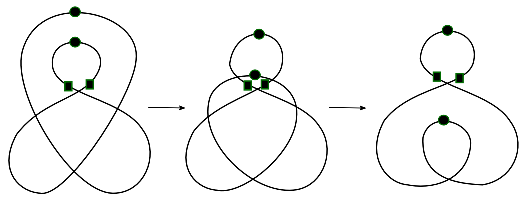


Figura 5: An example of plane curve.

2 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We begin with the basic curve K_0. In general, the family of such curves is also known as Lissajous curves. A general parametrization of this family is

$$\gamma(t) = (u\sin(at+\delta), v\sin(bt)),$$

where u, v, a, b and δ are real values. Visually, the ratio a/b determines the number of "lobes"of the figure. Therefore, for the Arnold basic curve without loss of generality we may assume that $a = 1$ and $b = 2$. We may abuse notation and denote the numerator of the curvature of γ by

$$\kappa(t,\delta) = -2uv\left(2sin(2t)\cos(t+\delta) - \sin(t+\delta)\cos(2t)\right).$$

Since $uv \neq 0$ so the roots of $\kappa(t,\delta)$ are $(t,\delta) = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$. This shows that the invariant $\iota(K_0) = 4$. The numerator of the first derivative of curvature is

$$\begin{aligned} &- uv\left(-8\sin(2t)\sin(t+\delta)\cos(2t)^2 - 12\cos(t+\delta)\cos(2t)^3 + 16\cos(2t)\cos(t+\delta)\right)v^2 \\ &+ \left(2\sin(2t)\cos(t+\delta)^2\sin(t+\delta) + 2\cos(t+\delta)^3\cos(2t) - \cos(2t)\cos(t+\delta)\right)u^2 \end{aligned}$$

The variables u and v do not affect geometry so without loss of generality we can assume $u = v = 1$. Putting $\delta = \pm\frac{\pi}{2}$ in the above expression and solving it we find that the vertex points happen at $t = 0$ and $t = \pi$. Therefore, $\nu(K_0) = 2$.

For the curve K_1 it is evident that $\iota(K_1) = 0$ and $\nu(K_1) = 4$. For the case of basic curves of type K_2, firstly it is trivial that $\iota(K_2) = 0$. Also due to Example 1.1, we have $0 < \nu(K_2) \leq 2$. For $i \geq 3$ we must pay attention to the fact that the curvature function at the vertex points in lobes has relative maximum. Thereupon, always between two relative maximum point of the curvature function there must exist at least one relative minimum. Now by using a same argument as given in the case K_2 and by induction on i, we obtain an upper bound for the invariant $\nu(K_i)$, i.e., we have $0 < \nu(K_i) \leq 2(i-1)$.

□

Proof of Theorem 1.2. The following figures illustrate five sequences of transitions on curves given in Figure 4. The next explanations describe what happens in each Figure.

(a) Starting by a curve of type K_1, the transition of second order inflection creates two inflections and a vertex between them. By passing through an inverse self-tangency, we obtain the curve of type D_2 with 4 vertices and 2 inflection points. Now we can perform another transition to vanish two vertices (Figure 6).

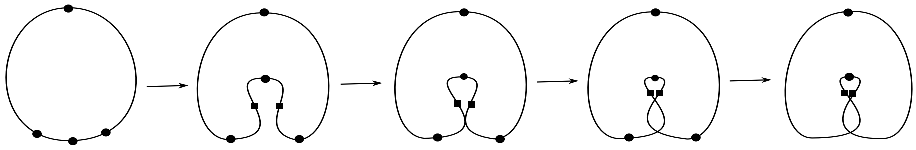


Figura 6: (a): Transition of a curve of type K_1 to D_2 ($\nu = 2$ and $\iota = 2$).

(b) In Figure 7 by using the item (a), we start by a curve of type D_2. Now we permit the curve to have two direct self-tangencies and obtain a curve of type D_{4_1} with 2 vertices and 2 inflection points.

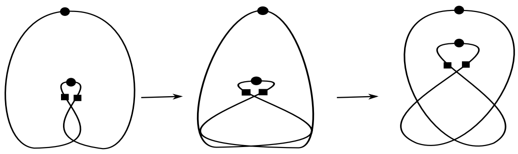


Figura 7: (b): Transition of a curve of type D_2 to D_{4_1} ($\nu = 2$ and $\iota = 2$).

(c) Beginning by a curve of type D_2 with 4 vertices and 2 inflections, using the geometric transition of second order vertex and then a transition of second order inflection, we have the birth of 2 new inflections and 2 new vertices. Then applying an inverse self-tangency, we gain a curve of type D_{4_2} with 6 vertices and 4 inflection points. Now the three free vertices can join and give us a curve with 4 vertices and 4 inflections (see Figure 8).

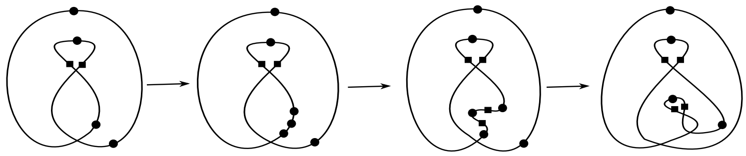


Figura 8: (c): Transition of a curve of type D_2 to D_{4_2} ($\nu = 4$ and $\iota = 4$).

(d) Using the second order vertex transition on a curve of type D_2, we have birth of 2 new vertices. Then, by performing transition of second order inflection we create 2 new inflections with 1 vertex point between them and apply an inverse self-tangency. Therefore, we obtain a curve of type D_{4_3} with 6 vertices and 6 inflections in total.

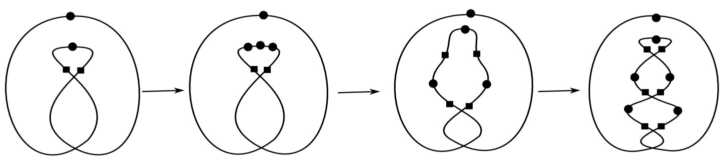


Figura 9: (d): Transition of a curve of type D_2 to D_{4_3} ($\nu = 6$ and $\iota = 6$).

(e) Finally, like item (c), starting by a curve of type D_2 with 4 vertices and 2 inflections and applying the transition of a second order vertex and a transition of second order inflection successively, we can create 3 new vertices and 2 new inflections. After performing an inverse self-tangency, one can obtain a curve of type D_{4_4} with 6 vertices and 4 inflections in total.

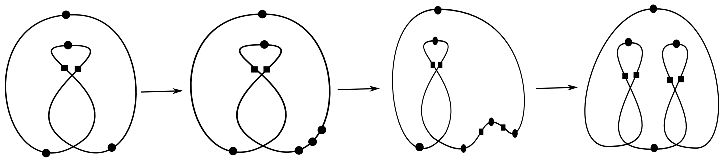


Figura 10: (e): Transition of a curve of type D_2 to D_{4_4} ($\nu = 4$ and $\iota = 4$).

□

3 Final considerations

Our study is part of an on going project on understanding geometric invariants of maps from closed surfaces in $\mathbb{R}^3$ to the plane. Theorem 1.2 is an essential key in the study of weighted graphs of these maps.

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