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Stabilization in semiactive systems using Sylvester equations: Approach

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Abstract. Active suspension systems for the vehicle control problem are formulated using output feedback. The present article proposes solving the output feedback control with eigenvalues for active suspension systems using Sylvester equations. The matrices are prescribed in advance and depend greatly on the controllability conditions, being assigned the system's observability eigenvalues. The theorems were presented using Sylvester equations. The algorithm was implemented based on Sylvester equation, and an example was presented with their conclusions.

Keywords. Sylvester equations, Eigenvalue assignment, Output feedback control, semi-active suspension systems.

1 Introduction

In moving mechanical systems, efforts are always made to reduce the influence of unwanted vibrations. In the case of automotive vehicles, the system responsible for such action is that of suspension. Not only in terms of comfort, but this system is necessary also for the safety and stability of the vehicle in maneuvers and when driving on terrains uneven. From the restrictions of projects with passive suspensions, whose parameters are fixed, and with the progress in technological and microelectronics development, as well as in the field of new actuators, active damper systems have

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evolved more and more, increasing the efficiency of these equipments and demanding, therefore, strategies of control of in order to provide an adequate action from the motion sensors. Allied to that, the development of control systems and its importance in the stability of systems mechanics (such as active suspension and control of flexible structures) motivate the study of efficient control techniques [4]. The semi-active system tries to vary the damping coefficient instead of a constant damping of the passive system. This is the main system that combines simplicity of control and quality in stabilization. The objective of this work is to develop the modeling of a control system of a semi-active damping system based on Sylvester's equations.

The remaining of the paper is organized in the following Sections: Section 2 in presented problem formulation. Section 4 Semi-active suspension systems is presented. Section 5 is presented Eigenstructure by Sylvester equation Sylvester. Section 6 is presented Numerical Algorithm of Eigenvalue Assignment by SOF (Static Output Feedback) and numerical example. Finally, Section 6 concludes the paper.

2 Problem formulation and control

Vehicle suspension plays a fundamental role concerning the level of passenger comfort. The mathematical model of a suspension system vehicle representing a quarter of a vehicle is used by most bibliographies available on the subject. The motion equation for the suspended mass is described by equation (1), where M_2 is the mass of a quarter car, k_f is the spring coefficient, d_s is the displacement of M_2 and d_r is the displacement of the wheel, spring and damper system.

$$M_2\ddot{d_s} = -k_f(d_s - d_r) - B_a(\dot{d_s} - \dot{d_r}) \tag{1}$$

The motion equation for the unsprung mass is described by equation (2), where M_1 is the mass of the system composed of the wheel, spring and damper.

$$M_1\ddot{d_r} = k_f(d_s - d_r) + B_a(\dot{d_s} - \dot{d_r}) - k_p(d_r - d_p) \tag{2}$$

Applying the state space model to Equation (1) and Equation (2), we arrive at the following state equation:

$$\begin{aligned} \dot{z}(t) &= A_p z(t) + B_p u(t) \\ y(t) &= Cz(t) \end{aligned} \tag{3}$$

with

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_f}{M_1} & \frac{-B_a}{M_1} & \frac{k_f}{M_1} & \frac{B_a}{M_1} \\ 0 & 0 & 0 & 1 \\ -\frac{k_f}{M_2} & \frac{B_a}{M_2} & \frac{-(k_p+k_f)}{M_1} & \frac{-B_a}{M_2} \end{bmatrix}$$

$$B_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_p}{M_2} \end{bmatrix}.$$

Let the the normal dynamical system (4)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{4}$$

Therefore, n dimensional matrices $A \in \mathbb{R}^{n\times n}$, state space $B \in \mathbb{R}^{n\times m}$ input matrix and $C \in \mathbb{R}^{p\times n}$ output matrix .

For the normal dynamical system (4), by choosing the following control law:

$$u(t) = -Ly(t) \tag{5}$$

with $L \in \mathbb{R}^{p\times n}$. It obtains the closed-loop system as follows:

$$\begin{aligned} \dot{x}(t) &= (A + BLC)x \\ y(t) &= Cx(t) \end{aligned} \tag{6}$$

Modern control is based on differential equations in the form of state space, and introduced to the control area from the 1960s onwards [1], [3]. Designing a controller for a dynamic system requires a model that represents the system's dynamic response. Most systems physics are complex and non-linear. Generally, the design needs to be based on a simplified but robust version of the model so that the control meets the performance requirements when applied to the actual device. In closed-loop control, the system includes a sensor to measure the output signal. The data acquired by the sensor is feedback to the controller that makes a comparison with the reference; the result of this comparison is the plant error. The controller sends a control effort signal to act on the process control variable from the error. A feature of both types of systems is the presence of the actuator. The actuator is a device that somehow influences the process control variable and receives the control effort signal directly from the controller in [3].

3 Semi-active suspension system

Active suspension systems the vehicle is described in [2]. Semi-Active Suspension Control Design for Vehicles is described in [4]. In a Semi-active suspension system, varying the damping coefficient of the shock absorber or spring constant through electronic control without inputting external energy to the system, except the device control that changes the damping coefficient or the elastic constant characterizes a semi-active suspension; mostly of the cases, the damping coefficient of the dissipative element in the system varies semi-active in [4]. In Figure (1), it is observed the semi-active suspension model. Where d_p, d_r and d_s represent the vertical displacements of the tire, wheel, and body, respectively. The spring is represented by its spring constant k_f and the damper by its coefficient B_a, while the constant k_p represents the tire elasticity. The masses of the body and wheel are respectively represented by M_1 and M_2.

Note that the semi-active suspension system is quite similar to the passive suspension system. The difference is that in the passive system the parameter B_a is constant, while in the semi-active system the B_a is variable, allowing the implementation of a control system to determine its value. Regarding the motion equations and state-space model of this example, both are the same as the passive suspension system. The only difference is that none of the parameters are changed in practice in the latter. In the semi-active suspension system, these parameters can be changed. Regarding the simulation, both the systems adjust the parameters in advance. Therefore, the simulation of both does not have considerable differences. There are several devices to implement a semi-active suspension system. In order to exemplify, here are three devices that are already used commercially: by the automotive industry: electro-hydraulic (EH), magnetorheological dampers (MR) and electrorheological (ER). EH dampers can be distinguished from traditional dampers due to the presence of electronic solenoid valves in your model, unlike of the conventional model that features only passive valves. These solenoid valves can vary the damping coefficient by changing the size of its orifices which, in turn, allow a greater or lesser flow of the fluid between the device's compression and traction chambers in [4].

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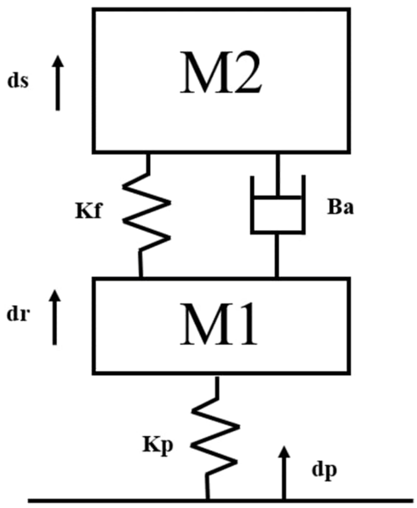

Figure 1: Model of semi-active suspension system of a car room.

The state equations, to obtain the state matrix A, the matrix input B , output matrix C , and forward transition matrix D.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{7}$$

4 Eigenstructure by Sylvester equation

Consider the following linear time-invariant descriptor system in [5].

$$\begin{aligned} E_s\dot{x}(t) &= A_s x(t) + B_s u(t) \\ y(t) &= C_s x(t) \end{aligned} \tag{8}$$

The Sylvester equations in [5].

$$A_sV_s - E_sV_sH_V = -B_sW_s, \qquad \sigma(H_V) \in \mathcal{C}^- \tag{9}$$

$$T_sA_s - H_TT_sE_s = -U_sC_s, \qquad \sigma(H_T) \in \mathcal{C}^- \tag{10}$$

$$T_sE_sV_s = 0 \tag{11}$$

Consider the $E_s = I_n$.

The system (4) can be written in the first-order state-space form (9), (10) (11). Thus, for obtaining the output feedback matrix K $\sigma(E_s, A_s + B_sKC_s) \in \mathcal{C}^-$, is used the Sylvester equation in [5].

The theorem 4.1 is based in [5], [7].

Theorem 4.1. *Let the system (8), be S-controllable, and $V_s \in \mathbb{R}^{n\times p}$ and $W_s \in \mathbb{R}^{m\times p}$ satisfy the equation (9). Then, the following hold.*

1) The matrices V_s and W_s given by (12),

$$\begin{bmatrix} A_s - \lambda_i E_s & B_s \\ T_s E_s & 0 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0 \; i = 1, 2, \cdots, q. \tag{12}$$

satisfy Sylvester matrix equation (9) for, $i = 1, 2, \cdots, q,$
2) When

$$rank(\begin{bmatrix} V_i \\ W_i \end{bmatrix}) = m \; i = 1, 2, \cdots, q. \tag{13}$$

hold, (11) gives all the solutions.

Proof. Based in [5]. □

Theorem 4.2. *Let* $E_s, A_s \in \mathbb{R}^{n\times n}$, $B_s \in \mathbb{R}^{n\times m}$, $H_V \in \mathbb{R}^{p\times p}$ *satisfy*

$$rank[A_s - \lambda E_s \; B_s] = n, \quad for\ any\ \lambda \in \sigma(H_V). \tag{14}$$

Further, let $H \in \mathbb{R}^{(n+m)\times m}[\lambda]$ *be a polynomial matrix satisfying* $[A_s - \lambda E_s \; B_s]H(\lambda) = 0_{n\times m}$.
Then:
(1) The matrices $V_s \in \mathbb{R}^{n\times p}$ *and* $W_s \in \mathbb{R}^{m\times p}$ *given by*

$$\begin{bmatrix} V_s \\ W_s \end{bmatrix} = Syl(H(\lambda), H_V, Z) \tag{15}$$

satisfy the matrix equation (9) for any matrix $Z \in \mathbb{R}^{m\times p}$. *(2) When* $rankH(\lambda) = r$ *for any* $\lambda \in \sigma(H_V)$, *all the matrices* V_s *and* W_s *satisfying the matrix equation (9) can be explicitly expressed by (15).*

Proof. Based in [5] and [6]. □

5 Numerical Algorithm

Let $G(\lambda)$ and X be defined as
$G(\lambda) = [A_s - \lambda E_s \; B_s]$; $X = \begin{bmatrix} V_s \\ W_s \end{bmatrix}$

The following basic procedure is proposed to calculate the feedback controller that stabilizes the closed loop system, when $m + p > n$. Closed loop eigenvalues are positioned arbitrarily close to the set ; they are symmetric sets of pre-specified eigenvalues. The (E_s, A_s, B_s, C_s) system is considered to be strongly controllable and strongly detectable.

Algorithm

Step 1: Choose an array $H_T \in \mathbb{R}^{q-p\times q-p}$ such that $\sigma(H_T) = \Lambda_T \in \mathcal{C}^-$ and Sylvester equation (10) is solved to find a matrix $T_s \in \mathbb{R}^{q-p\times n}$ such that

$$rank\ \left(\begin{bmatrix} T_s E_s \\ C_s \end{bmatrix}\right) = q \tag{16}$$

Step 2: Sylvester equation (9) is solved, for some $H_V \in \mathbb{R}^{p\times p}$ matrix such that $\sigma(H_V) = \Lambda_V \in \mathcal{C}^-$ taking into account that the matrix V_s must check the condition of the coupling (11) and taking into account that $rank\,(E_s V_s) = p$ .The condition (16) guarantees, in particular, that $rank(T_s E_s) = q$.

Step 3: By construction, the matrix V_s must verify that rank $(C_sV_s) = p$ and the matrix K can be calculated by:

$$K \quad = \quad W_s(C_sV_s)^{-1} \tag{17}$$

$\diamond$

Remark 5.1. *steps* 1 *and* 2 *can be solved using standard techniques for positioning the self-structure. Similar to the previous case, the matrices V_s and W_s used for the calculation of K can be constructed only with real elements. In particular: if $\lambda_i \mathcal{C}^-$, it is considered $\lambda_{i+1} = \lambda_i^*$ e* $\left\{ \begin{array}{ll} V_i = Re\,(v_i), & V_{i+1} = Imag\,(v_i) \\ W_i = Re\,(w_i), & W_{i+1} = Imag\,(w_i) \end{array} \right.$ *, where V_i and W_i denote the columns of the matrices V_s and W_s, respectively.*

In step 1, under the condition that the system is strongly observable (detectable). As will be seen later, degrees of freedom existing in the choice of V_s that satisfy the coupling condition $T_sE_sV_s = 0$, can also be used to guarantee obtaining K such that $KC_sV_s = W_s$ in [5].

5.1 Example

Consider a simple linear dynamical system (4) with $M_1 = 40$; $M_2 = 450$; $B_a = 1.2$; $k_f = 30$; $k_p = 160$; in the matrizes A, B, C :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_f}{M_1} & \frac{-B_a}{M_1} & \frac{k_f}{M_1} & \frac{B_a}{M_1} \\ 0 & 0 & 0 & 1 \\ -\frac{k_f}{M_2} & \frac{B_a}{M_2} & \frac{-(k_p+k_f)}{M_1} & \frac{-B_a}{M_2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{M_1} \\ 0 & 0 \\ \frac{k_p}{M_2} & \frac{1}{M_2} \end{bmatrix};$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Thus the matrizes A, B, C are following:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.75 & -0.03 & 0.75 & 0.03 \\ 0 & 0 & 0 & 1 \\ 0.0667 & 0.0027 & -0.4222 & -0.0027 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & -0.0250 \\ 0 & 0 \\ 0.3556 & 0.0022 \end{bmatrix};$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Algorithm basic

Resolved the equation (9) for calculate the matrices W_s, V_s, satisfies the equation (17) and the matrix K, such that $KC_sV_s = W_s$:

$T_s = \begin{bmatrix} -0.3851 & 0.3970 & -0.9512 & -0.0095 \end{bmatrix}$

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$$V_s = \begin{bmatrix} -0.0045 & -0.0023 & -0.0014 & -0.0014 \\ 0.0090 & 0.0069 & 0.0055 & 0.0055 \\ 0.0057 & 0.0039 & 0.0030 & 0.0030 \\ -0.0113 & -0.0118 & -0.0119 & -0.0119 \end{bmatrix}$$

$$W_s = \begin{bmatrix} 0.0650 & 0.0985 & 0.1314 & 0.1641 \\ 0.9978 & 0.9950 & 0.9912 & 0.9864 \end{bmatrix}$$

$K = \begin{bmatrix} -134.4978 & -177.1450 & -41.1356 \\ -35.5740 & -20.1672 & -83.9791 \end{bmatrix}$ where the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$, $\lambda_4 = -8.8859$.

6 Conclusions

This paper proposes solving the output feedback control problem with eigenvalues for a semi-active suspension system using Sylvester equations. Thus, the matrices formulated are simply prescribed in advance and highly dependent on the controllability and observability conditions of the system where the eigenvalues are assigned. The numerical method presented applies to the control design of the semi-active suspension system with multiple inputs and outputs.

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