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# Trapped depression solitary waves for the forced fifth-order forced Korteweg-de Vries equation

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**Abstract.** In this work, we investigate numerically trapped depression solitary waves in gravity-capillary flows for the fifth-order forced Korteweg-de Vries equation. We compute depression solitary waves with a single local minimum and three local minima that remain trapped bouncing back and forth between two topographic obstacles for large times. Besides, we study the wave stability of these trapped waves by disturbing their amplitudes. The depression solitary wave with a single local minimum is stable, whereas the one with three local minima splits into several depression solitary waves after a series of rebounds between the obstacles.

**Keywords.** Gravity-capillary waves, Solitary waves, Trapped waves

## 1 Introduction

The fifth-order forced Korteweg-de Vries equation (fKdV)

$$u_t + fu_x - \frac{3}{2}uu_x + \frac{b}{2}u_{xxx} - \frac{1}{90}u_{xxxxx} = \frac{1}{2}h_x(x), \tag{1}$$

is usually used to describe the dynamic of gravity-capillary flows over obstacles of small amplitudes [3, 9, 11]. In this equation $u(x,t)$ represents the free surface elevation, $h(x)$ the topography, $f$ and $b$ are small parameters related to the critical values of the Froude number ($F = 1$) and Bond number ($B = 1/3$) defined as

$$F = \frac{U_0}{\sqrt{gh_0}} \text{ and } B = \frac{\sigma}{\rho g h_0^2}, \tag{2}$$

respectively [9]. Here, $U_0$ is the background flow speed, $g$ is the acceleration of gravity, $h_0$ is the depth in the far field of the obstacle, $\sigma$ is the coefficient of the surface tension and $\rho$ is the constant density of the fluid.

A phenomenon that has recently captured attention of many researchers in the field of water waves is the mechanism known as trapped waves. The concept of trapped waves is mainly featured by waves that remain bouncing back and forth in certain region of space, usually an obstacle, for large times. In the past few years, many authors have investigated trapped waves in gravity flows [5–8]. These authors considered the third-order fKdV equation to investigate trapped waves as perturbations of steady solutions of the third-order fKdV equation. More recently, considering gravity-capillary flows, Flamarion and Ribeiro-Jr observed that depression trapped waves can be spontaneously generated in the fKdV [3] and in the full Euler model [4].

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In this work, we compute numerically different types of depression solitary waves for the unforced problem (1) ($h_x = 0$) and find regimes such that these waves bounce back and forth between two bumps remaining trapped for large times. In addition, the wave stability of these trapped waves is investigated. The novelty is twofold: (i) we compute trapped depression solitary waves with a single local minimum and with three local minima; (ii) we investigate the wave stability of these trapped waves. The study of the stability reveals that the wave with a single local minimum is stable with respect to perturbations of its amplitude while the depression solitary waves with three local minima splits into different depression solitary waves after a series of rebounds.

This article is organized as follows. In section 2 we present the numerical methods. The results are presented in section 3 and the conclusion in section 4.

# 2 Numerical methods

Depression solitary waves are computed for the unforced problem ($h_x = 0$) with $f = 0$ through the Newton method's type in the Fourier space by solving the equation

$$\Big(-c-\frac{b}{2}k^2-\frac{1}{90}k^4\Big)\widehat{u}-\frac{3}{4}\widehat{u^2}=0. \tag{3}$$

The initial guess for the wave profile is defined as

$$u_0(x) = -a_0\exp\Big(-x^2/w\Big)\cos(k_0x), \tag{4}$$

where $a_0$ is the amplitude of the initial wave, $w$ is its width and $k_0$ is the wave number. The initial guess for the wave speed is taken as the linear phase speed given by the dispersion relation of the unforced problem with $f = 0$

$$c_0 = -\frac{1}{2}bk_0^2-\frac{1}{90}k_0^4. \tag{5}$$

The solution can be continued in the amplitude by using the prior converged solution of the Newton's method (3) as the initial guess to obtain a new solution with a larger amplitude $a$ and new speed $c$.

The fifth-order fKdV equation (1) is solved numerically using a Fourier pseudospectral method in the same fashion as done by Flamarion et al. [2]. The initial data is always taken as a depression solitary wave computed using the Newton's method type described above. A sketch of the physical problem is depicted in Figure 1.

The computational domain is periodic with a uniform grid with $N$ points and step $\Delta x$. The spatial derivatives are computed spectrally [10], and the time evolution is computed using the Runge-Kutta fourth-order method (RK4) with time step $\Delta t$. The obstacles are modeled by the function

$$h(x) = 0.001\Big[\exp\Big(-(x+25)^2\Big)+\exp\Big(-(x-25)^2\Big)\Big]. \tag{6}$$

Typical computations are carried out using the following set of parameters: $\Delta x = 0.04$, $N = 2^{13}$, $L = N\Delta x/2 = 200$ and $\Delta t = 0.01$. For the initial guess we fix $a_0 = 0.001$, $k_0 = 15\pi/L$ and $w = 10^{-3}$. The perturbations of the Froude and Bond numbers are fixed as $f = -c - 0.01$ and $b = 1/30$ respectively.

# 3 Results

## 3.1 Trapped waves

Figure 2 shows two types of depression solitary waves computed through the Newton's method. The shape profile of depression solitary wave in Figure 2 (left) is qualitatively similar to the trapped

 

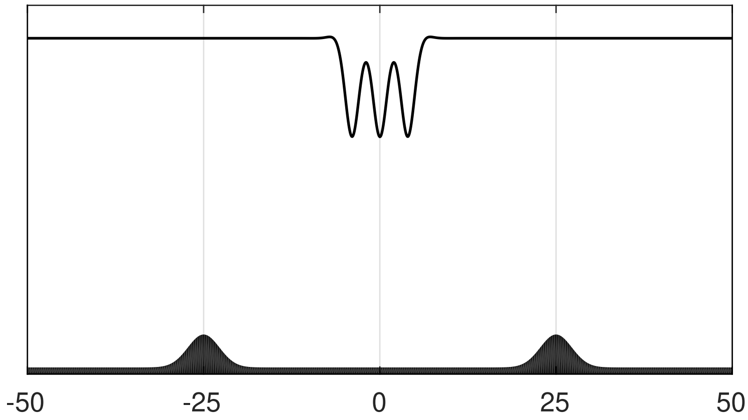


Figure 1: Sketch of the physical problem.

depression wave reported in [3, 4]. However, the wave depicted in Figure 2 (right) has not yet been contemplated in the literature as a trapped wave.

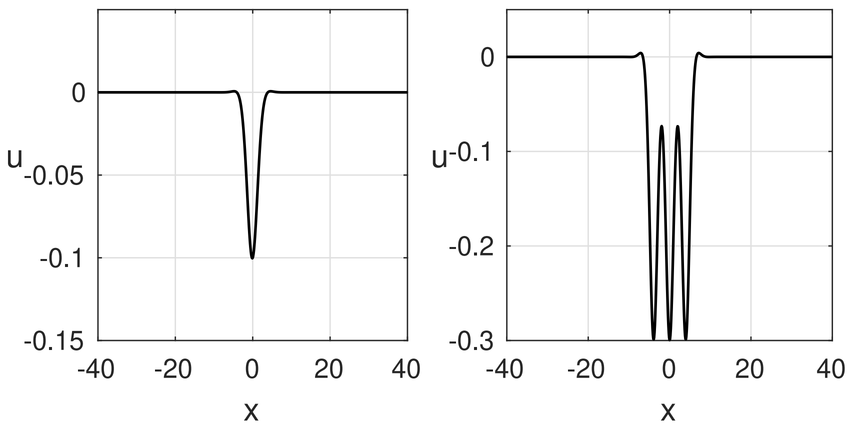


Figure 2: Depression solitary waves with different shapes. Left: depression solitary wave with amplitude $a = 0.1003$. Right: depression solitary wave with amplitude $a = 0.2994$.

Figure 3 displays the evolution of the two types of depression solitary waves depicted in Figure 2. As we can see, these waves bounce back and forth between the obstacles remaining trapped for large times. Only later, they accumulated enough energy to overcome the obstacle and then escape out. As far as we know, there are no theory to describe this behaviour for the fifth-order fKdV equation.

## 3.2 Wave stability

The wave stability of trapped waves has been investigated by many authors [1, 6–8] In this section we investigated the wave stability of the trapped waves computed in the previous section. For the wave with a single local minimum a recently study was done by Flamarion [1] in the presence of an accelerated moving forcing. The author showed that this wave is stable for small perturbations of its amplitude in the sense that this wave still remains trapped for large times.

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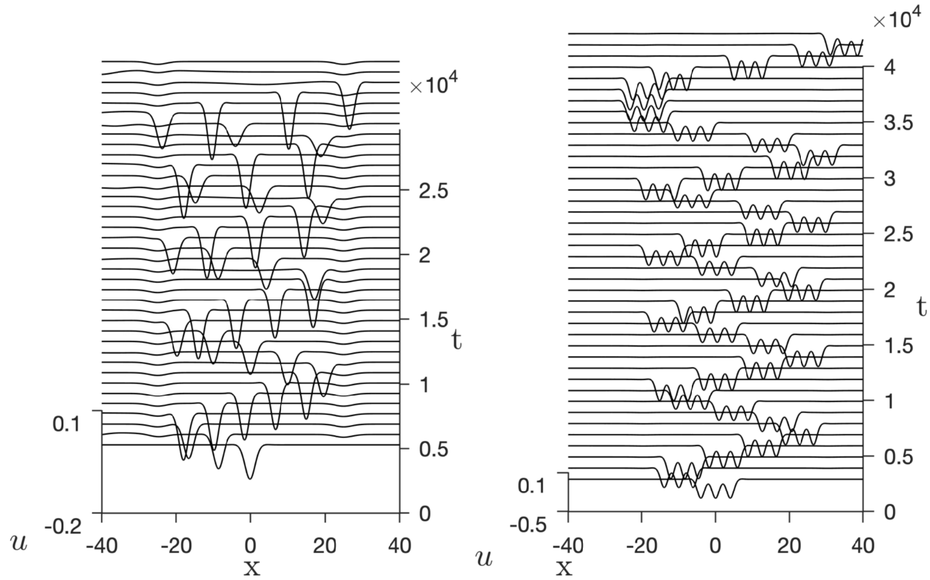


Figure 3: The evolution of two depression solitary waves with different shapes. Left: depression solitary wave of Figure 2 (left). Right: depression solitary wave of Figure 2 (right).

In order to carry out our study, we initially consider the wave with a single local minimum ($u_s$) and analyse the evolution of the initial data $u_0 = \alpha u_s$ through equation (1), where $\alpha \approx 1$. Figure 4 (left) displays the time of escape of the disturbed wave as a function of $\alpha$. As we can see, the wave is stable with respect to the parameter $\alpha$ in the sense that the wave remains trapped between the obstacles for large times. However, we see no pattern in the amplitude of the escaping wave right after it escapes out (see Figure 4 (right)).

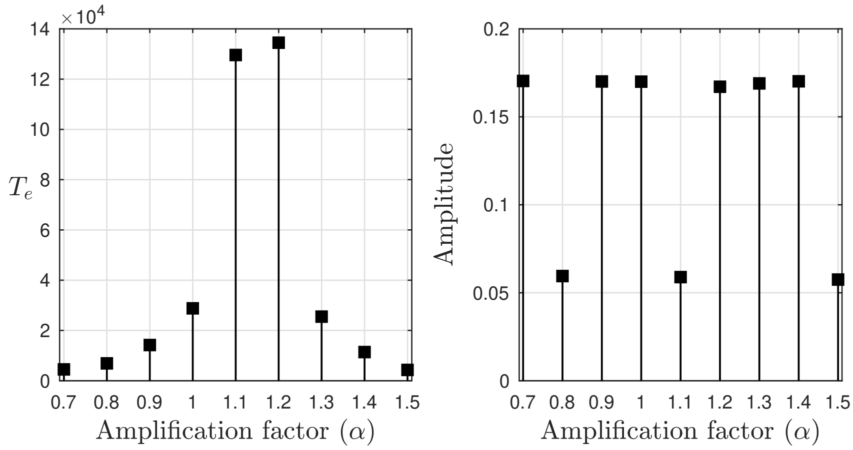


Figure 4: The time of escape of the disturbed wave $\alpha u_s$ as a function of $\alpha$. Right: The amplitude of the escaping wave as a function of $\alpha$ at time $t = T_e$.

Different from the previous case, the wave stability study for the the wave with three local

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minima unveils an interesting phenomenon. After a few rebounds this wave splits into several depression solitary waves with a single local minimum. Figure 5 displays the evolution of the wave $u_0 = \beta u_m$, where $\beta = 1.1$ and $u_m$ is the wave displayed in Figure 2 (right). As we can see the wave initially bounces back and forth with its shape almost unchangeable, then this wave splits into two depression solitary waves, one with a single local minimum and one with two local minima. Later, the wave with a single local minima escapes out of the obstacle and the remained wave splits into two depression waves with only a single local minimum, which remain trapped between the obstacles.

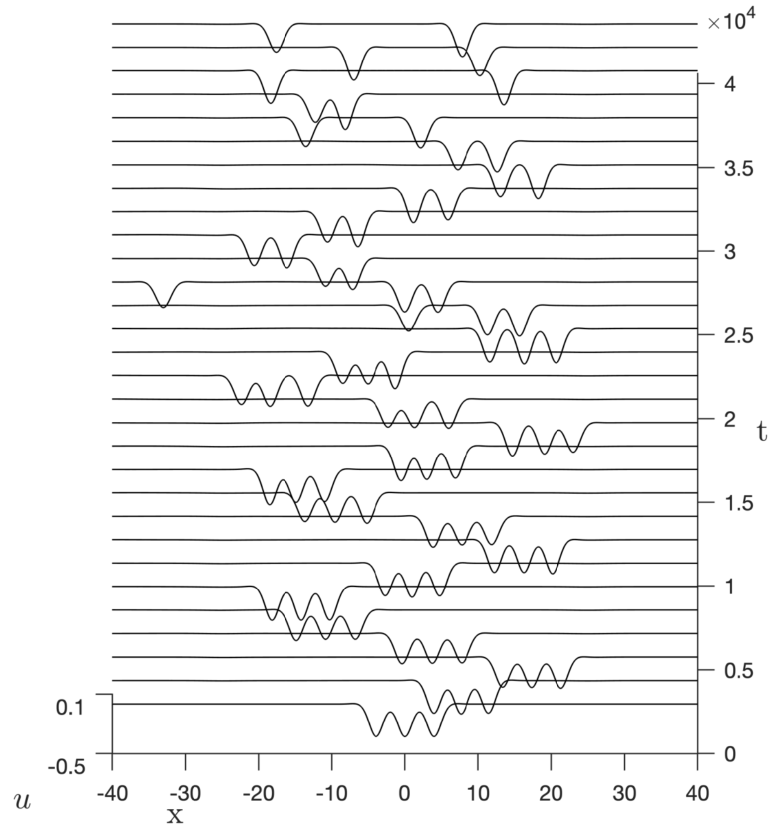


Figure 5: The evolution of the perturbed depression solitary wave with three local minima.

# 4 Conclusion

In this article, we have found numerically new types of depression solitary waves that can be trapped between two bumps in gravity-capillary flows for large times. We showed that depression solitary waves with a single local minimum are stable with respect to perturbations of their amplitudes. In addition, we showed that depression solitary waves with three local minima splits into different depression solitary waves after a series of rebounds.

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## References

[1] M.V. Flamarion. "Generation of trapped depression solitary waves in gravity-capillary flows over an obstacle". In: **Computational and Applied Mathematics** 41 (2022), pp. 1–9. DOI: 10.1007/s40314-021-01734-w.

[2] M.V. Flamarion, Milewski P. A., and A. Nachbin. "Rotational waves generated by current-topography interaction". In: **Studies in Applied Mathematics** 142 (2019), pp. 433–464. DOI: doi.org/10.1111/sapm.12253.

[3] M.V. Flamarion and R. Ribeiro-Jr. "Gravity-capillary flows over obstacles for the fifth-order forced Korteweg-de Vries equation". In: **Journal of Engineering Mathematics** 129 (2021), pp. 1–10. DOI: 10.1007/s10665-021-10153-z.

[4] M.V. Flamarion and R. Ribeiro-Jr. "Gravity-capillary wave interactions generated by moving disturbances: Euler equations framework". In: **Journal of Engineering Mathematics** 132 (2022), pp. 1–10. DOI: 10.1007/s10665-021-10207-2.

[5] R. Grimshaw and M. Malewoong. "Transcritical flow over obstacles and holes: forced Korteweg-de Vries framework". In: **Journal of Fluid Mechanics** 881 (2019), pp. 660–678. DOI: 10.1017/jfm.2019.767.

[6] H. Kim and H. Choi. "A study of wave trapping between two obstacles in the forced Korteweg-de Vries equation". In: **Journal of Engineering Mathematics** 108 (2017), pp. 197–208. DOI: 10.1007/s10665-017-9919-5.

[7] S. Lee. "Dynamics of Trapped Solitary Waves for the Forced KdV Equation". In: **Symmetry** 21 (2018), pp. 467–490. DOI: 10.3390/sym10050129.

[8] S. Lee and S. Whang. "Trapped supercritical waves for the forced KdV equation with two bumps". In: **Applied Mathematical Modeling** 39 (2015), pp. 2649–2660. DOI: 10.1016/j.apm.2014.11.007.

[9] P. A. Milewski and J-M. Vanden-Broeck. "Time dependent gravity-capillary flows past an obstacle". In: **Wave Motion** 29 (1999), pp. 63–79. DOI: 10.1016/S0165-2125(98)00021-3.

[10] L.N. Trefethen. **Spectral Methods in MATLAB**. Philadelphia: SIAM, 2001. ISBN: 0898714656.

[11] Y. Zhu. "Resonant generation of nonlinear capillary-gravity waves". In: **Physics of Fluids** 7 (1995), pp. 2294–2296. DOI: 10.1063/1.868479.

 