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On Generalizations of the Initial and Terminal Value Theorems.

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Abstract. The Initial and Terminal Value Theorems provide information about the limiting values of applications whose Laplace Transform is known. Such theorems, in addition to being relevant in their original form, are susceptible to generalizations that are also important. This article demonstrates the Initial and Terminal Value Theorems and their generalizations, studying these results for possible applications in Engineering and Physics. The novelty of this study is mainly in the presentation of the results. Despite being a review article, it presents proofs of theorems that are uncommon to be found in the literature. Therefore, this work contributes in the form of a complementary material for the study of the Laplace Transform and its applications.

Key-words. Initial and Terminal Value Theorems, Laplace Transform, Differential Equations.

1 Introduction

The Initial and Terminal Value Theorems have great relevance to differential equations. There are important applications in Electrical Engineering, mainly in the analysis of electrical circuits [4]. In order to find the limiting values of functions that model a circuit, the Laplace Transform is used, transforming the function with a domain in t into another with a domain in s. In this way, the Initial and Terminal Value Theorems allow us to find the limiting values of these respective functions without calculating their Inverse Laplace Transform, which considerably reduces the algebraic operability, in addition to providing information about the function as $t \to 0$ or as $t \to \infty$, via the Laplace Transform, even though the function is not known explicitly [7].

There are applications in statistical mechanics in stochastic processes, such as Brownian motion and the Kubo relation [1]. The Initial and Terminal Value Theorems are used to verify the asymptotic behavior of the diffusion constant. Additionally, the concept of normal diffusion was studied by analyzing the correlation function in its asymptotic format [1]. In this case, the generalizations of the Initial and Terminal Value Theorems were widely used.

Finally, it is possible to find in the literature some materials that exemplify applications of such theorems and their generalizations in works on Engineering and Physics [1, 3–6]. On the other hand, a mathematical proof of these results is difficult to find in the literature, especially for generalizations. Therefore, this work aims to demonstrate these theorems and contribute as a complementary study material on Laplace Transform and its applications.

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2 Preliminary Concepts

In this section, we present some definitions and fundamental theorems for the understanding and proof of the Initial and Terminal Value Theorems and their generalizations.

2.1 Function of Exponential Order γ

Definition 2.1. *A function f has exponential order γ if there exist constants $M > 0$ and $\gamma > 0$ such that for some $t_0 \geq 0$*

$$|f(t)| \leq Me^{\gamma t}, \qquad \forall\, t \geq t_0. \tag{1}$$

2.2 Piecewise Continuity

Definition 2.2. *A function f has a jump discontinuity at a point t_0 if both the limits*

$$\lim_{t \to t_0^-} f(t) = \alpha \qquad \textit{and} \qquad \lim_{t \to t_0^+} f(t) = \beta \tag{2}$$

exist (as numbers) and are distinct, that is, $\alpha \neq \beta$.

Definition 2.3. *A function f is piecewise continuous on the interval $[0, \infty[$ if*

1) $\lim_{t \to 0^+} f(t) = f(0)$;

2) f is continuous on the interval $[0, x[$, for all $x > 0$, except, possibly, at a finite number of points $\tau_1, \tau_2, \ldots, \tau_n$ in $[0, x[$, at which f has a jump discontinuity.

2.3 Integral Transforms

Definition 2.4. *Given a function $f : I \subset \mathbb{R} \longrightarrow \mathbb{C}$, where I is a real interval, an integral transform of f is a function defined by*

$$F(s) = \int_I K(s,t) f(t) dt \tag{3}$$

where $F(s)$ is called the integral transform of the function f, $K(s,t)$ is called the kernel of the transform, and s is a real or complex parameter.

2.4 The Laplace Transform

Definition 2.5. *Let $f : [0, \infty[\subset \mathbb{R} \longrightarrow \mathbb{C}$ a function and s a real or complex parameter. We define the Laplace Transform of f as:*

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{4}$$

when the improper integral is convergent.

Note that the Laplace Transform is an integral transform with kernel $K(s,t) = e^{-st}$.

Theorem 2.1. *Let f of exponential order γ and piecewise continuous on $[0, x]$ for all $x > 0$. Then $F(s) = \mathcal{L}(f(t))$ is convergent for $Re(s) > \gamma$.*

A proof for this theorem can be found in [7].

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2.5 Admissible Functions

Definition 2.6. *We define $\mathbb{L}$ as the set of all functions of type $f : [0, \infty[\subset \mathbb{R} \longrightarrow \mathbb{C}$ such that the Laplace transform exists for some value of s.*

Definition 2.7. *A function $f : [0, \infty[\subset \mathbb{R} \longrightarrow \mathbb{C}$ is admissible if it is piecewise continuous on $[0, x]$ for all $x > 0$ and has exponential order γ.*

By the Theorem 2.1, admissible functions belong to $\mathbb{L}$. However, there are certainly functions in $\mathbb{L}$ that do not satisfy one or both of the admissibility conditions.

Theorem 2.2. *If $f \in \mathbb{L}$ and $\mathcal{L}(f(t)) = F(s)$ then*

$$\lim_{Re(s) \to \infty} F(s) = 0 \tag{5}$$

A proof for this theorem can be found in [2].

2.6 Transform of the First Derivative

Theorem 2.3. *Let f a differentiable function of exponential order γ, such that f' is piecewise continuous on $[0, x]$ for all $x > 0$. Then*

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0) \tag{6}$$

for $Re(s) > \gamma$.

Proof. Since

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\, dt = \lim_{u \to \infty} \left(\int_0^u e^{-st} f'(t)\, dt \right)$$

using integration by parts

$$\int_0^u e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^u + s \int_0^u e^{-st} f(t)\, dt = e^{-su} f(u) - f(0) + s \int_0^u e^{-st} f(t)\, dt$$

Applying the limit,

$$\lim_{u \to \infty} \left(e^{-su} f(u) - f(0) + s \int_0^u e^{-st} f(t)\, dt \right) = s \int_0^\infty e^{-st} f(t)\, dt - f(0)$$

for $Re(s) > \gamma$ because, thus, $(e^{-su} f(u)) \to 0$ according to $u \to \infty$.

We have

$$\int_0^\infty e^{-st} f'(t)\, dt = s \int_0^\infty e^{-st} f(t)\, dt - f(0)$$

for $Re(s) > \gamma$, that is,

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

□

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3 Initial and Terminal Value Theorems and Generalizations

In this section, we present and demonstrate the Initial and Terminal Value Theorems, as well as their important generalizations.

3.1 Initial Value Theorem

Theorem 3.1. *Let f a differentiable function of exponential order γ, such that f' is piecewise continuous on $[0, x]$ for all $x > 0$ and $F(s) = \mathcal{L}(f(t))$. Then,*

$$f(0) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) \qquad (s\, real)$$

Proof. According to Theorems 2.2 and 2.3, when $s > \gamma$

$$\lim_{s\to\infty} (sF(s) - f(0)) = 0. \tag{7}$$

Implying

$$f(0) = \lim_{s\to\infty} s\,F(s). \tag{8}$$

Since f is piecewise continuous, $\lim_{t\to 0^+} f(t) = f(0)$ which concludes the proof. □

3.2 Terminal Value Theorem

Theorem 3.2. *Let f a differentiable function of exponential order γ, such that f' is piecewise continuous on $[0, x]$ for all $x > 0$ and suppose $\lim_{t\to\infty} f(t)$ exists. Then, the value of this limit is given by*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s) \qquad (s\, real) \tag{9}$$

where $F(s) = \mathcal{L}(f(t))$.

Proof. f being of exponential order γ, by the Theorem 2.3, for $s > \gamma$

$$\mathcal{L}(f'(t)) = sF(s) - f(0). \tag{10}$$

Applying the limit,

$$\begin{aligned} \lim_{s\to 0} (sF(s) - f(0)) &= \lim_{s\to 0} \int_0^\infty e^{-st}\, f'(t)\, dt \\ &= \int_0^\infty f'(t)\, dt \\ &= \lim_{u\to\infty} \int_0^u f'(t)\, dt \\ &= \lim_{u\to\infty} (f(u) - f(0)) \,. \end{aligned} \tag{11}$$

In this way, we get

$$\lim_{s\to 0} (sF(s) - f(0)) = \lim_{u\to\infty} (f(u) - f(0)) \implies \lim_{u\to\infty} f(u) = \lim_{s\to 0} sF(s). \tag{12}$$

□

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3.3 Generalization of the Initial Value Theorem

Theorem 3.3. *Let f and g differentiable functions of exponential order γ, with piecewise continuous derivatives on $[0, x]$ for all $x > 0$ where $F(s) = \mathcal{L}(f(t))$ and $G(s) = \mathcal{L}(g(t))$, where s real parameter. If $\lim_{t\to 0^+} \frac{f(t)}{g(t)} = 1$, then $\lim_{s\to\infty} \frac{F(s)}{G(s)} = 1$.*

Proof.

$$\lim_{s\to\infty} \frac{F(s)}{G(s)} = \lim_{s\to\infty} \frac{sF(s)}{sG(s)} = \frac{\lim\limits_{s\to\infty} sF(s)}{\lim\limits_{s\to\infty} sG(s)} \overset{Theo.\ 3.1}{=} \frac{\lim\limits_{t\to 0^+} f(t)}{\lim\limits_{t\to 0^+} g(t)} = \lim_{t\to 0^+} \frac{f(t)}{g(t)} = 1.$$

□

3.4 Generalization of the Terminal Value Theorem

Theorem 3.4. *Let f and g differentiable functions of exponential order γ, with piecewise continuous derivatives on $[0, x]$ for all $x > 0$ where $F(s) = \mathcal{L}(f(t))$ and $G(s) = \mathcal{L}(g(t))$, where s real parameter. If $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$, then $\lim_{s\to 0} \frac{F(s)}{G(s)} = 1$.*

Proof.

$$\lim_{s\to 0} \frac{F(s)}{G(s)} = \lim_{s\to 0} \frac{sF(s)}{sG(s)} = \frac{\lim\limits_{s\to 0} sF(s)}{\lim\limits_{s\to 0} sG(s)} \overset{Theo.\ 3.2}{=} \frac{\lim\limits_{t\to\infty} f(t)}{\lim\limits_{t\to\infty} g(t)} = \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

□

4 An Interesting Application

The book [8] brings up two interesting exercises at the end of chapter 1 that can be used as examples of the application of the Initial and Terminal Value Theorems. We will transform such exercises, which deal with the Gamma Function, in the following theorem.

The Gamma Function was first studied by Euler, in 1730, in research on a way to interpolate the factorial of a number. It was later studied by other mathematicians, including Adrian Marie Legendre, who, in 1809, named it as we know it today.

We denote by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx \tag{13}$$

the Gamma Function, which converges when $p > 0$ (real). This function has several interesting properties. It satisfies, for example, the recurrence relation $\Gamma(p + 1) = p\Gamma(p)$. It also satisfies $\Gamma(n) = (n - 1)!$ for $n \in \mathbb{N}$.

It can be easily demonstrated, using Definition 2.5, that $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ for $n \in \mathbb{N}$. We can generalize such a transform, for not natural powers, using the Gamma Function. In fact,

$$\mathcal{L}(t^\nu) = \int_0^\infty t^\nu e^{-st}\, dt \tag{14}$$

by a change of variables, where $x = st$ with $s > 0$, we have

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$$\mathcal{L}\left(t^{\nu}\right) = \int_0^{\infty} e^{-x}\left(\frac{x}{s}\right)^{\nu}\frac{1}{s}\,dx$$
$$= \frac{\displaystyle\int_0^{\infty} x^{\nu}e^{-x}\,dx}{s^{\nu+1}}. \tag{15}$$

So, we get

$$\mathcal{L}\left(t^{\nu}\right) = \frac{\Gamma(\nu+1)}{s^{\nu+1}}, \qquad \nu > -1, \qquad s \in \mathbb{R}_{+}^{*}. \tag{16}$$

With this, we can work on the exercises mentioned in the book [8], transforming them into the following theorem.

Theorem 4.1. *Let f an exponential order differentiable function γ, with piecewise continuous derivatives on $[0, x]$ for all $x > 0$ where $F(s) = \mathcal{L}\left(f(t)\right)$, with s a real parameter and c a random real constant.*

(a) If $\lim\limits_{t\to 0}\dfrac{f(t)}{ct^p} = 1$, *for* $p > -1$, *then* $\lim\limits_{s\to\infty}\dfrac{F(s)}{\frac{c\Gamma(p+1)}{s^{p+1}}} = 1$;

(b) If $\lim\limits_{t\to\infty}\dfrac{f(t)}{ct^p} = 1$, *for* $p > -1$, *then* $\lim\limits_{s\to 0}\dfrac{F(s)}{\frac{c\Gamma(p+1)}{s^{p+1}}} = 1$.

Proof. First, note that $\dfrac{c\Gamma(p+1)}{s^{p+1}} = \mathcal{L}(ct^p)$, from Equation (16). Then,

(a)

$$\lim_{s\to\infty}\frac{F(s)}{\frac{c\Gamma(p+1)}{s^{p+1}}} = \lim_{s\to\infty}\frac{sF(s)}{s\left(\frac{c\Gamma(p+1)}{s^{p+1}}\right)} = \frac{\lim\limits_{s\to\infty} sF(s)}{\lim\limits_{s\to\infty} s\left(\dfrac{c\Gamma(p+1)}{s^{p+1}}\right)} \overset{Theo.\ 3.1}{=} \frac{\lim\limits_{t\to 0} f(t)}{\lim\limits_{t\to 0} ct^p} = \lim_{t\to 0}\frac{f(t)}{ct^p} = 1$$

(b)

$$\lim_{s\to 0}\frac{F(s)}{\frac{c\Gamma(p+1)}{s^{p+1}}} = \lim_{s\to 0}\frac{sF(s)}{s\left(\frac{c\Gamma(p+1)}{s^{p+1}}\right)} = \frac{\lim\limits_{s\to 0} sF(s)}{\lim\limits_{s\to 0} s\left(\dfrac{c\Gamma(p+1)}{s^{p+1}}\right)} \overset{Theo.\ 3.2}{=} \frac{\lim\limits_{t\to\infty} f(t)}{\lim\limits_{t\to\infty} ct^p} = \lim_{t\to\infty}\frac{f(t)}{ct^p} = 1$$

□

5 Final Remarks

This text exposed the importance of the Initial and Terminal Value Theorems and their generalizations for differential equations, especially for Laplace Transforms. Such results have also been proven, which is uncommon to find in the literature (especially the generalizations).We hope that this brief work contributes in the form of a complementary material for the study of Laplace Transform and its applications.

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