Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 9, n. 1, 2022.

Trabalho apresentado no XLI CNMAC, Unicamp - Campinas - SP, 2022.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics
Preprint

On the paths through which the fire progress faster

Hengameh R. Dehkordi[1]
UFABC, Santo André, SP

Abstract. Usually, there are some paths through which the fire spreads faster in wildfire propagation. These paths are called strategic paths here as they play an essential role in improving the fire fighting management strategies. In this work, we find the equations of such paths and the fire fronts' equation. We also calculate the area consumed by the fire. Moreover, implementing the approach in the MATLAB environment, we provide an example to demonstrate the efficiency of our proposed results.

Palavras-chave. Randers metric, strategic paths, wildfire propagation, fire front

1 Introduction

Every year the wildfires become harsher, and more complex mathematics is needed to study their behavior more efficiently and to present a more reliable model for their propagation. Randers geometry is a strong tool for investigating complex phenomena and has been recently used to analyze waves propagation and wildfire spreading [2–6]. In [2] it is shown that the paths of fire particles are geodesics of the Randers metric associated with the propagation. The authors found the Randers metric from the equation of an ellipse on the space. They determined the ellipse equation from the experimental data and it depends on the wind and fuel characteristics, such as density, humidity, temperature, etc. Here, we apply the Randers geometry techniques to study fire propagation in a flat terrain under the influence of the wind.

For the given wildfire, there is a Riemannian metric h associated with the propagation which is determined from the ellipse equation. In fact, in the cases that we verify in this work, one does not need find the Randers metric associated with the propagation and it suffices to find the Riemannain metric. We assume that the wind is constant and mild, $h(W, W) < 1$, and the terrain across which the fire is spreading is an open subset of $\mathbb{R}^2$. Although, the results of this work are valid for every dimension n we focus on the dimension 2 for the potential applications.

It is note that, the study of wildfire propagation from the Randers geometry point of view has just begun, and to the best of our knowledge no work has been done regarding finding the paths through which the fire spread faster. We call these paths the strategic paths. A wave ray is the path of a fire particle in the propagation. In this work, we find the equations of fire fronts and strategic paths and also calculate the area consumed by the fire at each time. By the fire front at each time, we mean the perimeter of the burnt area. We furnish the work with an example, implemented in the MATLAB programming language, in which a hypothetical wildfire spreads. We consider several cases for the wind and the fuel properties.

1.1 Preliminaries

We recall several geometric objects that we need to establish our main results. For more details, see [8]. Let M be a smooth manifold of dimension 2, for instance an open subset of $\mathbb{R}^2$,

[1] hengameh.r@ufabc.edu.br

$p = (x, y) \in M$ a point and T_pM the set of vectors tangent to M at p. Assume that $V = (u, v) \in T_pM$ is a vector according to the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ for T_pM. A *Riemannian metric* on M is a smooth function h such that to each point $p \in M$, h assigns a positive-definite inner product $h_p : T_pM \times T_pM \to \mathbb{R}$. Given any Riemannian metric h and a smooth vector field W on M such that $h(W, W) < 1$, the function $F : TM \to \mathbb{R}$ given by

$$F(V) = \frac{\sqrt{h^2(W,V) + \lambda h(V,V)}}{\lambda} - \frac{h(W,V)}{\lambda}, \tag{1}$$

where $\lambda = 1 - h(W, W)$ is called a *Randers metric* [1] and (M, F) the Randers space. A smooth curve in a Randers space is called a *geodesic* if it is locally the shortest time path connecting any two nearby points on this curve. Given a Randers space (M, F), the Randers geodesics are solutions of

$$\frac{d^2x^r}{dt^2} + \frac{1}{2}\sum_{l,k,j=1}^{2} g^{rl}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^l}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}, \qquad r = 1, 2, \tag{2}$$

where $[g_{rj}] = \frac{1}{2}\frac{\partial^2 F^2}{\partial v_r \partial v_j}$, $[g^{rj}]$ is the inverse matrix of $[g_{rj}]$, $(v_1, v_2) = (u, v)$, and $(x^1, x^2) = (x, y)$ [8]. Given a Riemannian space (M, h), the Riemannain geodesics are solutions of

$$\frac{d^2x^r}{dt^2} + \frac{1}{2}\sum_{l,k,j=1}^{2} h^{rl}\left(\frac{\partial h_{lk}}{\partial x^j} + \frac{\partial h_{lj}}{\partial x^k} - \frac{\partial h_{kj}}{\partial x^l}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}, \qquad r = 1, 2, \tag{3}$$

where $[h^{rj}]$ is the inverse matrix of $[h_{rj}]$ and $(x^1, x^2) = (x, y)$ [7].

For a given wildfire propagation, there is an ellipse associated to it given below

$$Q(u, v) = \left(\frac{u\cos\theta - v\sin\theta}{a}\right)^2 + \left(\frac{u\sin\theta + v\cos\theta}{b}\right)^2 = 1, \tag{4}$$

where θ is the rotational angle around the origin, and a and b are the ellipse axes determined from the experimental data [2]. Once we have the ellipse equation, the Riemannian metric is

$$h = HessQ = \frac{1}{2}[Q_{v_1v_2}], \tag{5}$$

where $(v_1, v_2) = (u, v)$ and Q_{v_i} is the partial derivative of Q with respect to v_i, $i = 1, 2$. From the Riemannian metric h one finds the associated Randers metric by using Eq. (1).

2 Main results and an example

Here, we assume that a wildfire spreads in a flat terrain across which the fuel, temperature, humidity, etc., are distributed smoothly. The constant wind W is spreading across the land. For each time τ, we provide the strategic path and fire front equations and the area consumed by the fire till time τ. The fire starts from a point that is considered the origin of the coordinate system.

The first step of applying Randers geometry techniques to analyze the fire behavior is finding the ellipse equation associated with the propagation. To find the ellipse, one assumes that the fuel is distributed uniformly and the other conditions, that is, humidity, temperature, and so on, are uniform across the space. Then, we determine the ellipse equation, which is an object in the tangent space. Since the fuel and other conditions vary smoothly across the space, the ellipse equation changes smoothly across the space. From this ellipse equation and using Eq. (5) we find the Riemannian metric. Next, one applies Theorem 3.2 and Proposition 3.2 of [2] to establish the following results.

If W is constant and satisfies

$$\sum_{r,j,k=1}^{2} W^k \frac{\partial h_{rj}}{\partial x^k} = 0, \tag{6}$$

then, assuming that $\alpha(t)$ is the solution of the Eq. (3) with the initial conditions $\alpha(0) = 0$ and $||\alpha'(0)||_h = 1$ we have that,

1. the fire front at time τ is $\{\tau W + \alpha(\tau)\}$,

2. the strategic path until time τ is $tW + \alpha(t)$, $t \in [0, \tau]$, provided that the Euclidean norm of vector $\overrightarrow{\tau W + \alpha(\tau)}$ is maximum among all curves $\alpha(t)$, and

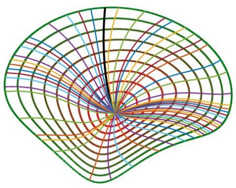
3. the consumed area by the fire after τ unit of time of propagation is the Euclidean area of closed curve $\{\tau W + \alpha(\tau)\}$.

Example 2.1. *We simulate the hypothetical wildfires which are spreading in different seasons in Pico da Lombada, located in the Ibitipoca State Park in the state of Minas Gerais. One tourism attraction of this park is long walks, in particular towards Pico da Lombada. The interesting characteristic of this place is the* $360°$ *view from there to the horizon; that is, one has a view to the horizon in all directions. We consider different cases for the fuel characteristics and the wind* W *that is blowing across Pico da Lombada. We show the strategic path and fire fronts and calculate the area consumed by the fire for each case.*

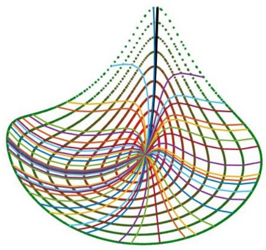
In Figs. 1a-1d, the northern wind with the magnitude $\frac{1}{3}km/h$ *is blowing, and the model of propagation is given from the beginning of the propagation till* 10 *hours later. The time interval between each fire front and the next one is* 1 *hour. Interestingly, the propagation model and the strategic path might differ for wildfire propagation in the same area and wind but with different fuel characteristics. We provided the information on the ellipse associated with the propagation for each case, given by the experimental data. The strategic path is shown with the thick black curve, and some of the wave rays are shown with other colors.*

Figs. 2a-2d show the propagation with the eastern wind of magnitude $\frac{1}{2}km/h$ *and the same conditions on the fuel, humidity, etc., for different moments after initiating the fire. Here, the time interval from each fire front to the next one is* 12 *minutes. The objective is to show how the shape of fire changes as time passes which means the propagation model is not predictable efficiently without applying mathematical techniques. The figures show the strategic paths of fire fronts and calculate the burnt area.*

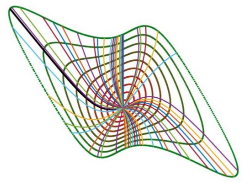
In Fig. 2a, it has been 2.5 *hours since the fire has started, and during this time, the fire consumes about* $0.01ha$. *The propagation model seems a translated ellipse and the strategic path is toward the northeastern. In Fig. 2b, the fire is burning for* 5 *hours and it consumes about* $0.3ha$. *The propagation model looks like an oval and the strategic path starts leaning toward the east. In Fig. 2c, the fire is burning for* 8 *hours and consumes almost* $2.2ha$. *As can be seen in the figure, the propagation shape has changed noticeably and the strategic path is toward the southeastern. Finally, in Fig. 2d, the fire is burning for* 10 *hours and is progressing toward the southeast harshly. Here, the fire consumes nearly* $4.6ha$.



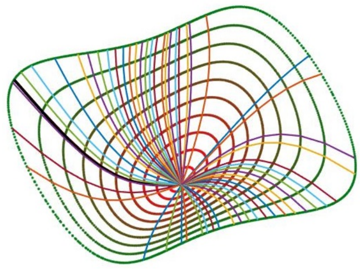
(a) With, $a = 2, b = 3$, and $\alpha = \frac{2}{5} + x$. The area consumed by the fire is $23ha$.



(b) With, $a = 1.2, b = 3.3$, and $\alpha = \frac{1}{5} - x$. The area consumed by the fire is $38.5ha$.

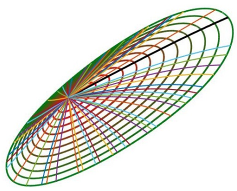


(c) With, $a = 2+x^2, b = 3$, and $\alpha = \pi/4$. The area consumed by the fire is $15ha$.

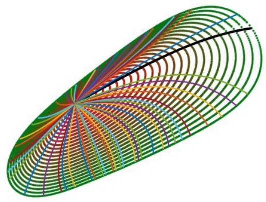


(d) With, $a = 1.2+x^2, b = 2$, and $\alpha = \pi/4$. The area consumed by the fire is $9.5ha$.

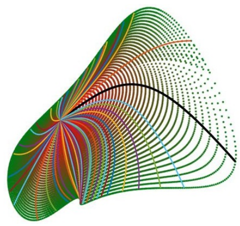
Figure 1: The fire propagation in *Ibitipoca State Park with the northern wind.*



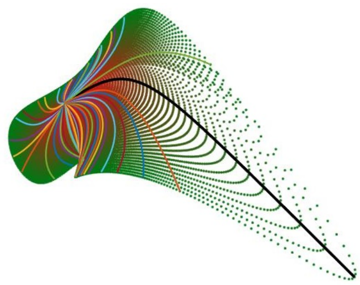
(a) With, $a = 1 + x^2, b = 3$, and $\alpha = \pi/4$. The area consumed by the fire is $0.01ha$.



(b) With, $a = 1 + x^2, b = 3$, and $\alpha = \pi/4$. The area consumed by the fire is $0.3ha$.



(c) With, $a = 1 + x^2, b = 3$, and $\alpha = \pi/4$. The area consumed by the fire is $2.2ha$.



(d) With, $a = 1 + x^2, b = 3$, and $\alpha = \pi/4$. The area consumed by the fire is $4.6ha$.

Figure 2: The fire propagation in *Ibitipoca State Park with the eastern wind.*

DOI: 10.5540/03.2022.009.01.0289

3 Conclusion and final remarks

In this work, by using Randers geometry methods, we studied the propagation model for the wildfire spread in a flat terrain under the influence of a constant wind. We provided the equations of the fire fronts and strategic paths, the paths along which the fire progress faster. Moreover, we calculated the area consumed by the fire. This way, the fire behavior is predicted more precisely and reliable. Therefore, the results of this work can be used in the fire fighting management strategies to reduce the losses caused by the fire. As an interesting problem for the future works, one can verify the wildfire propagation under the influence of a time-dependent wind.

References

[1] D. Bao, C. Robles, and Z. Shen. "Zermelo navigation on Riemannian manifolds". In: **J. Differ. Geom.** 66.3 (2004), pp. 377–435.

[2] H. R Dehkordi. "Applications of Randers geodesics for wildfire spread modelling". In: **Appl. Math. Model** (2022).

[3] H. R. Dehkordi. "Mathematical modeling of wildfire propagation in an agricultural land". In: **Proc. Ser. Braz. Soc. Comput. Appl. Math.** 8.1 (2021).

[4] H. R. Dehkordi and A. Saa. "Huygens' envelope principle in Finsler spaces and analogue gravity". In: **Class. Quantum Grav.** 36.8 (2019), p. 085008.

[5] G. W. Gibbons and C. M. Warnick. "The geometry of sound rays in a wind". In: **Contemporary Physics** 52.3 (2011), pp. 197–209.

[6] S. Heefer, C. Pfeifer, and A. Fuster. "Randers p p-waves". In: **Physical Review D** 104.2 (2021), p. 024007.

[7] J. M Lee. **Introduction to Riemannian manifolds**. Springer International Publishing, New York, 2018.

[8] Z. Shen. **Lectures on Finsler geometry**. World Scientific, Singapore, 2001.

DOI: 10.5540/03.2022.009.01.0289