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Calculation of Green's function for Poisson's equation on a semi-disk using a Fourier transform in the radial variable

Roberto Toscano Couto[1]
UFF, Niterói, RJ

Abstract. A new calculation of Green's function for the problem with Poisson's equation on a semi-disk under mixed Dirichlet-Neumann boundary conditions is presented. The method consists in first (a) employing a Fourier transform in the radial variable to calculate the solution of the simpler problem that is obtained with the homogenization of the boundary conditions, and then (b) inferring the desired Green's function by comparing the expression of this calculated solution with the one given by Green's formula. The solution that the method yields is elaborated to the point of having the same closed form that the method of images provides.

Keywords. Green's function, Poisson, semi-disk, radial, Fourier transform

1 Introduction

This work aims to present a new method for calculating the Green's function for the problem depicted in Figure 1, that is, for Poisson's equation when the problem domain Ω is the semi-disk shown in that figure and the boundary conditions are those indicated there: Dirichlet's at the base and Neumann's on the circumference. This problem is formulated as follows:

$$\begin{cases} \nabla^2 u(r,\theta) = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = h(r,\theta)\,,\ r \in (0,b)\,,\ \theta \in (0,\pi)\,. \\ u(r,0) = f_0(r)\,,\ r \in [0,b]\,;\ u(r,\pi) = f_\pi(r)\,,\ r \in (0,b]\,;\ \dfrac{\partial u}{\partial r}(b,\theta) = g(\theta)\,,\ \theta \in (0,\pi)\,. \end{cases} \tag{1}$$

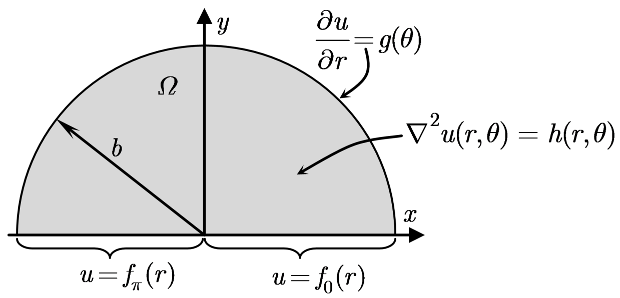

Figure 1: The boundary value problem for which Green's function is calculated.

The Green's function for this problem is found in Ref. [1], where it is calculated by the method of imagens; it is given by eqs. (24) and (26) in that bibliographic reference:

$$G(r,\theta|r',\theta') = \frac{1}{2}\ln\frac{r^2 + r'^2 - 2rr'\cos(\theta'+\theta)}{r^2 + r'^2 - 2rr'\cos(\theta'-\theta)} + \frac{1}{2}\ln\frac{r^2 r'^2 + b^4 - 2b^2 rr'\cos(\theta'+\theta)}{r^2 r'^2 + b^4 - 2b^2 rr'\cos(\theta'-\theta)} \quad . \tag{2}$$

[1]rtoscano@id.uff.br

Therefore, a specific objective of ours is to derive this Green's function again, but by the method presented in this work. In doing this, we reach the broader goal of fully explaining the method.

Section 2 describes the main steps of the method. Section 3 presents the application of the method to calculate the Green's function for problem (1). Section 4 ends the body of the paper with final comments.

2 Description of the method applied to problem (1)

The method developed in this work takes advantage of the fact that the Green's function of problem (1) does not depend on the functions f_0, f_π, and h. Then, to calculate it, we may use the following problem, which is a simplified version of problem (1), in which all boundary conditions were homogenized:

$$\begin{cases} \nabla^2 v(r,\theta) = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = h(r,\theta)\,,\ r \in (0,b)\,,\ \theta \in (0,\pi)\,. \\ v(r,0) = v(r,\pi) = 0\,,\ r \in (0,b]\,;\ \dfrac{\partial v}{\partial r}(b,\theta) = 0\,,\ \theta \in (0,\pi)\,. \end{cases} \tag{3}$$

a) The first step of the method is the calculation of the solution v to this problem. It is convenient to use the variable ρ related to r as follows:

$$r = b\,e^{-\rho} \in [0,b] \quad \Leftrightarrow \quad \rho = -\ln(r/b) \in [0,\infty)\ . \tag{4}$$

Using the chain rule, we have

$$v(r,\theta) = v(b\,e^{-\rho},\theta) \equiv V(\rho,\theta) \quad \Rightarrow \quad r\frac{\partial v}{\partial r}(r,\theta) = -\frac{\partial V}{\partial \rho}(\rho,\theta) \quad \text{and} \quad r^2\frac{\partial^2 v}{\partial r^2} = \frac{\partial^2 V}{\partial \rho^2} + \frac{\partial V}{\partial \rho}\ .$$

Consequently, Poisson's equation in (3) takes the simpler form

$$r^2\nabla^2 v(r,\theta) = \frac{\partial^2 V}{\partial \rho^2} + \frac{\partial^2 V}{\partial \theta^2} = r^2 h(r,\theta) \equiv H(\rho,\theta)\ ,$$

and the homogeneous boundary conditions become

$$V(\rho,0) = v(r,0) = 0 \quad \text{and} \quad \frac{\partial V}{\partial \rho}(\rho,\theta)\Big|_{\rho=0} = \Big[-r\frac{\partial v}{\partial r}(r,\theta)\Big]_{r=b} = 0\ .$$

Therefore, problem (3) in terms of the variable ρ becomes

$$\begin{cases} \dfrac{\partial^2 V}{\partial \rho^2} + \dfrac{\partial^2 V}{\partial \theta^2} = H(\rho,\theta)\,,\ \rho \in (0,\infty)\,,\ \theta \in (0,\pi)\,. \\ V(\rho,0) = V(\rho,\pi) = 0\,,\ \rho \in [0,\infty)\,;\ \dfrac{\partial V}{\partial \rho}(0,\theta) = 0\,,\ \theta \in (0,\pi)\,. \end{cases} \tag{5}$$

This problem is represented in Figure 2.

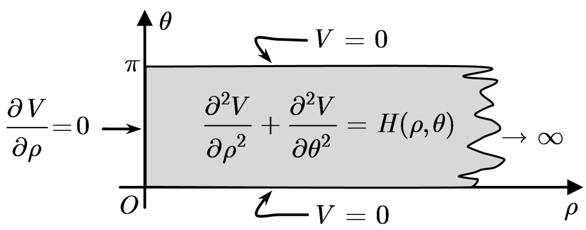

Figure 2: The problem in (5).

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We see that, in the plane of ρ and θ, the problem domain takes the shape of a semi-infinite slab; this fact and the homogeneous Neumann condition on the boundary at $\rho = 0$ justifies the use of the following cosine Fourier transform to solve problem (5) for its solution $V(\rho, \theta)$:

$$\mathcal{F}_c\{V(\rho,\theta)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty V(\rho,\theta) \cos k\rho \, d\rho \;\equiv\; \bar{V}(k,\theta) \;\; . \tag{6}$$

b) The second step is the determination of the Green's function for problem (1) from the solution $v(r,\theta)\big[= V(\rho,\theta)\big]$ of problem (3) calculated in the first step. Let us see how.

Since $v(\vec{r}) = -\frac{1}{2\pi} \iint_\Omega G(\vec{r}\,|\vec{r}\,')\, h(\vec{r}\,')\, dA'$ {cf. Ref. [2], eq. (1.42), which is here adapted to two dimensions}, or, in terms of plane polar coordinates,

$$v(r,\theta) = -\frac{1}{2\pi} \int_0^\pi d\theta' \int_0^b dr'\, r'\, h(r',\theta')\, G(r,\theta|r',\theta') \;\; ,$$

we see a possibility of inferring an expression for $G(r,\theta|r',\theta')$ by writing the already calculated solution $v(r,\theta)$ in the form of the double integral on the right side of the above equation. Actually, it is better to perform this step using the ρ variable, that is, using the calculated $V(\rho,\theta)$ instead of $v(r,\theta)$. In this variable the above equation becomes (we omit the variables of Green's function, simply denoting it by G, regardless of whether it is a function of r or ρ)

$$V(\rho,\theta) \;=\; -\frac{1}{2\pi} \int_0^\pi d\theta' \int_\infty^0 [-d\rho' \underbrace{r']\, r' h(r',\theta')}_{H(\rho',\theta')}\, G \;=\; -\frac{1}{2\pi} \int_0^\pi d\theta' \int_0^\infty d\rho'\, H(\rho',\theta')\, G \;\; . \tag{7}$$

We will see that this writing is not an automatic task, requiring some artifices in the first step in order to put $\bar{V}(k,\theta)$ with a form that leads to that double integral.

c) A third step is still necessary to deduce (2), because in the second step we obtain an integral representation for the Green's function G. We need, therefore, to evaluate an integral to get G in the closed form of (2).

3 Application of the method to calculate the Green's function for problem (1)

Now we apply the method to calculate the Green's function of problem (1). We begin by solving problem (3) employing the cosine Fourier transform defined in (6). Using it to transform the partial differential equation in (5) (use is made of the formula for $\mathcal{F}_c\{f''\}$, given, for example, in Ref. [3], sec. 7.6) and taking into account the boundary condition at $\rho = 0$, we obtain

$$-k^2\bar{V}(k,\theta) - \sqrt{\frac{2}{\pi}} \overbrace{\frac{\partial V}{\partial \rho}(0,\theta)}^{0} + \frac{d^2\bar{V}}{d\theta^2} = \mathcal{F}_c\{H(\rho,\theta)\} \equiv \bar{H}(k,\theta) \;\Rightarrow\; \frac{d^2\bar{V}}{d\theta^2} - k^2\bar{V}(k,\theta) = \bar{H}(k,\theta) \;\; . \tag{8}$$

We can solve this nonhomogeneous ordinary differential equation by using variation of parameters {cf. Ref. [4], sec. 1.9}. Since the general solution of the associated homogeneous equation is

$$\bar{V}_H(k,\theta) = c_1 \cosh k\theta + c_2 \sinh k\theta \;\; ,$$

the form of a particular solution is given by

$$\bar{V}_P(k,\theta) = A(\theta) \cosh k\theta + B(\theta) \sinh k\theta \;\; , \tag{9}$$

where the functions $A(\theta)$ and $B(\theta)$ must be solutions of the system of equations

$$\begin{cases} A' \cosh k\theta + B' \sinh k\theta = 0 \\ kA' \sinh k\theta + kB' \cos k\theta = \bar{H} \end{cases}$$

Solving it, we get

$$A'(\theta) = -\frac{\bar{H}\sin k\theta}{k} \quad \Rightarrow \quad A(\theta) = -\frac{1}{k}\int_0^\theta \bar{H}(k,\theta')\sinh k\theta \, d\theta' \ ,$$

$$B'(\theta) = \frac{\bar{H}\cosh k\theta}{k} \quad \Rightarrow \quad B(\theta) = \frac{1}{k}\int_0^\theta \bar{H}(k,\theta')\cosh k\theta \, d\theta' \ .$$

With the substitution of these results into (9), we obtain

$$\begin{aligned} \bar{V}_P(k,\theta) &= -\frac{\cosh k\theta}{k}\int_0^\theta \bar{H}(k,\theta')\sinh k\theta' d\theta' + \frac{\sinh k\theta}{k}\int_0^\theta \bar{H}(k,\theta')\cosh k\theta'\theta' \\ &= -\frac{1}{k}\int_0^\theta \bar{H}(k,\theta')\sinh k(\theta'-\theta)\, d\theta' \ . \end{aligned}$$

The general solution $\bar{V}_H(k,\theta) + \bar{V}_P(k,\theta)$ of (8) is then given by

$$\bar{V}(k,\theta) = c_1\cosh k\theta + c_2 \sinh k\theta - \frac{1}{k}\int_0^\theta \bar{H}(k,\theta')\sinh k(\theta'-\theta)\, d\theta' \ , \tag{10}$$

or, equivalently, with the lower limit of integration replaced by π,

$$\bar{V}(k,\theta) = d_1\cosh k\theta + d_2 \sinh k\theta - \frac{1}{k}\int_\pi^\theta \bar{H}(k,\theta')\sinh k(\theta'-\theta)\, d\theta' \ . \tag{11}$$

To determine c_1, c_2, d_1, and d_2, we impose the two homogeneous Dirichlet conditions in (5). To this end, we apply the cosine Fourier transform to them, obtaining

$$\bar{V}(k,0) = \bar{V}(k,\pi) = 0 \ . \tag{12}$$

Imposing these two conditions on the expression of $\bar{V}(k,\theta)$ which is given by (10), we get

$$\bar{V}(k,0) = \underline{c_1 = 0} \quad \Rightarrow \quad \bar{V}(k,\pi) = c_2\sinh k\pi - \frac{1}{k}\int_0^\pi \bar{H}(k,\theta')\sinh k(\theta'-\pi)\, d\theta' = 0$$
$$\Rightarrow \quad c_2 = \frac{1}{k\sinh k\pi}\int_0^\pi \bar{H}(k,\theta')\sinh k(\theta'-\pi)\, d\theta' \ ,$$

whose substitution into (10) gives

$$\bar{V}(k,\theta) = \frac{\sinh k\theta}{k\sinh k\pi}\int_0^\pi \bar{H}(k,\theta')\sinh k(\theta'-\pi)\, d\theta' + \frac{1}{k}\int_0^\theta \bar{H}(k,\theta')\sinh k(\theta-\theta')\, d\theta' \ ,$$

or

$$\begin{aligned} \bar{V}(k,\theta) \ &= \ \frac{1}{k\sinh k\pi}\int_0^\pi d\theta' \, \bar{H}(k,\theta')\sinh k\theta \sinh k(\theta'-\pi) \\ &+ \frac{1}{k\sinh k\pi}\int_0^\theta d\theta' \, \bar{H}(k,\theta')\sinh k\pi \sinh k(\theta_> - \theta_<) \ , \end{aligned} \tag{13}$$

where we used the definition

$$\theta_< \ (\theta_>) = \text{the smaller (larger) of } \theta \text{ and } \theta' \ . \tag{14}$$

Imposing now the two conditions in (12) on equation (11), we obtain

$$\bar{V}(k,0) = d_1 - \frac{1}{k}\int_\pi^0 \bar{H}(k,\theta')\sinh k\theta'\,d\theta' \quad \Rightarrow \quad d_1 = -\frac{1}{k}\int_0^\pi \bar{H}(k,\theta')\sinh k\theta'\,d\theta' \ ;$$

$$\bar{V}(k,\pi) = d_1\cosh k\pi + d_2\sinh k\pi \quad \Rightarrow \quad d_2 = \frac{\cosh k\pi}{k\sinh k\pi}\int_0^\pi \bar{H}(k,\theta')\sinh k\theta'\,d\theta' \ .$$

The substitution of the two results into (11) gives

$$\begin{aligned}\bar{V}(k,\theta) \ &= \ -\frac{\cosh k\theta}{k}\int_0^\pi \bar{H}(k,\theta')\sinh k\theta'\,d\theta' + \frac{\sinh k\theta\cosh k\pi}{k\sinh k\pi}\int_0^\pi \bar{H}(k,\theta')\sinh k\theta'\,d\theta' \\ &\quad + \frac{1}{k}\int_\theta^\pi \bar{H}(k,\theta')\sinh k(\theta'-\theta)\,d\theta' \ ,\end{aligned}$$

or

$$\begin{aligned}\bar{V}(k,\theta) \ &= \ \frac{1}{k\sinh k\pi}\int_0^\pi d\theta'\bar{H}(k,\theta')\big[-\sinh k\pi\cosh k\theta\sinh k\theta' + \sinh k\theta\cosh k\pi\sinh k\theta'\big] \\ &\quad + \frac{1}{k\sinh k\pi}\int_\theta^\pi d\theta'\,\bar{H}(k,\theta')\sinh k\pi\sinh k(\theta_> - \theta_<) \ .\end{aligned} \tag{15}$$

Equations (13) and (15) are two equivalent expressions for the cosine Fourier transform of the solution $V(\rho,\theta)$ to problem (5), but they are not suitable to express $V(\rho,\theta)$ in the form of the double integral in (7). The reason is that in this double integral the interval of the integration with respect to θ' is $[0,\pi]$, and those two expressions contain integrations with respect to θ' over intervals other than $[0,\pi]$. However, if we add (13) and (15) we get another equivalent expression which does have the proper form:

$$\begin{aligned}&(2k\sinh k\pi)\,\bar{V}(k,\theta) = \\ &\int_0^\pi d\theta'\bar{H}(k,\theta')\big[\sinh k\theta\sinh k(\theta'-\pi) - \sinh k\pi\cosh k\theta\sinh k\theta' + \sinh k\theta\cosh k\pi\sinh k\theta'\big] \\ &+ \int_0^\pi d\theta'\,\bar{H}(k,\theta')\sinh k\pi\sinh k(\theta_> - \theta_<) \ ,\end{aligned}$$

or

$$\bar{V}(k,\theta) = \int_0^\pi d\theta'\,\bar{H}(k,\theta')\frac{\Gamma(k,\theta,\theta')}{2k\sinh k\pi} \ , \tag{16}$$

where

$$\begin{aligned}\Gamma(k,\theta,\theta') \ \equiv \ &\sinh k\theta\sinh k(\theta'-\pi) - \sinh k\pi\cosh k\theta\sinh k\theta' + \sinh k\theta\cosh k\pi\sinh k\theta' \\ &+ \sinh k\pi\sinh k(\theta_> - \theta_<) \ .\end{aligned} \tag{17}$$

We now take the inverse cosine Fourier transform of (16), and also substitute the defining expression for $\bar{H}(k,\theta') = \mathcal{F}_c\{H(\rho',\theta'\}$, obtaining

$$\begin{aligned}
V(r,\theta) &= \sqrt{\frac{2}{\pi}}\int_0^\infty dk\,\{\bar{V}(k,\theta)\}\cos k\rho = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\left\{\int_0^\pi d\theta'\,\left[\bar{H}(k,\theta')\right]\frac{\varGamma(k,\theta,\theta')}{2k\sinh k\pi}\right\}\cos k\rho \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty dk\left\{\int_0^\pi d\theta'\left[\sqrt{\frac{2}{\pi}}\int_0^\infty d\rho'\,H(\rho',\theta')\cos k\rho'\right]\frac{\varGamma(k,\theta,\theta')}{2k\sinh k\pi}\right\}\cos k\rho \\
&= -\frac{1}{2\pi}\int_0^\pi d\theta'\int_0^\infty d\rho'\,H(\rho',\theta')\underbrace{\int_0^\infty dk\,\frac{-2\varGamma(k,\theta,\theta')\cos k\rho\cos k\rho'}{k\sinh k\pi}}_{G}\quad .
\end{aligned}$$

By comparing this result with (7), we infer that Green's function is as indicated above by G.

However, this inferred expression for Green's function is different from the one in (2). To match them, it is necessary to evaluate the above integral with respect to k. But before that let us write (17) in terms of $\theta_<$ and $\theta_>$ only, that is, without the explicit presence of θ or θ':

$$\begin{aligned}
\varGamma(k,\theta,\theta') &= \sinh k\theta\,[\sinh k\theta'\cosh k\pi - \sinh k\pi\cosh k\theta'] - \sinh k\pi\cosh k\theta\sinh k\theta' \\
&\quad + \sinh k\theta\cosh k\pi\sinh k\theta' + \sinh k\pi\,[\sinh k\theta_>\cosh k\theta_< - \sinh k\theta_<\cosh k\theta_>] \\
&= \sinh k\theta\sinh k\theta'\cosh k\pi - \sinh k\theta\sinh k\pi\cosh k\theta' - \sinh k\pi\cosh k\theta\sinh k\theta' + \sinh k\theta\cosh k\pi\sinh k\theta' \\
&\quad + \sinh k\pi\sinh k\theta_>\cosh k\theta_< - \sinh k\pi\sinh k\theta_<\cosh k\theta_> \\
&= 2\sinh k\theta\sinh k\theta'\cosh k\pi - \sinh k\pi(\sinh k\theta\cosh k\theta' + \cosh k\theta\sinh k\theta') \\
&\quad + \sinh k\pi\sinh k\theta_>\cosh k\theta_< - \sinh k\pi\sinh k\theta_<\cosh k\theta_> \\
&= 2\sinh k\theta\sinh k\theta'\cosh k\pi - \sinh k\pi\sinh k(\theta+\theta') + \sinh k\pi\sinh k\theta_>\cosh k\theta_< - \sinh k\pi\sinh k\theta_<\cosh k\theta_> \\
&= 2\sinh k\theta_<\sinh k\theta_>\cosh k\pi - \sinh k\pi\sinh k(\theta_> + \theta_<) + \sinh k\pi\sinh k\theta_>\cosh k\theta_< - \sinh k\pi\sinh k\theta_<\cosh k\theta_> \\
&= 2\sinh k\theta_<\sinh k\theta_>\cosh k\pi - \sinh k\pi(\cancel{\sinh k\theta_>\cosh k\theta_<} + \sinh k\theta_<\cosh k\theta_>) \\
&\quad + \sinh k\pi\,\cancel{\sinh k\theta_>\cosh k\theta_<} - \sinh k\pi\sinh k\theta_<\cosh k\theta_> \\
&= 2\sinh k\theta_<\sinh k\theta_>\cosh k\pi - 2\sinh k\pi\sinh k\theta_<\cosh k\theta_> \;=\; -2\sinh k\theta_<\sinh k(\pi-\theta_>)\quad .
\end{aligned}$$

$$\therefore\quad G = \int_0^\infty \frac{4\sinh k\theta_<\sinh k(\pi-\theta_>)\cos k\rho\cos k\rho'}{k\sinh k\pi}\,dk\quad . \tag{18}$$

Let us obtain G in the form of (2) from the above result. We start with an usual application of the residue theorem. Writing the above integral (of an even function) in the complex k plane as half of its extension to the whole k-axis, closing the path with a circumference of radius which goes to ∞, and evaluating the residues at the simple poles of the integrand inside the closed contour, that is, at the zeros $n\mathrm{i}$ $(n = 1,2,3,\cdots)$ of $k\sinh k\pi$ ($k=0$ is a removable singularity), that theorem allows us to write

$$\begin{aligned}
G &= \frac{1}{2}\cdot 2\pi\mathrm{i}\sum_{n=1}^{\infty}\lim_{k\to \mathrm{i}n}\frac{k-\mathrm{i}n}{\sinh k\pi}\;\frac{2\cos k\rho\cos k\rho'\;2\sinh k\theta_<\sinh k(\pi-\theta_>)}{k} \\
&= \pi\mathrm{i}\sum_{n=1}^{\infty}\frac{1}{\pi\cos n\pi}\;\frac{[2\cosh n\rho\cosh n\rho']\,[2\mathrm{i}\sin n\theta_<\cdot\mathrm{i}\sin n(\pi-\theta_>)]}{\mathrm{i}n} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\left(\frac{r}{r'}\right)^n + \left(\frac{r'}{r}\right)^n + \left(\frac{rr'}{b^2}\right)^n + \left(\frac{b^2}{rr'}\right)^n\right]2\sin n\theta_<\sin n(\pi-\theta_>)\ ,
\end{aligned}\tag{19}$$

where in the last step we used (4) to go back to the variables r and r'.

Now we calculate the sum of this infinite series. To this end, we establish the following formula:

$$S_-(R,B) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\Big(\frac{R}{B}\Big)^n 2\sin n\phi' \sin n\phi = \frac{1}{2}\ln\frac{R^2+B^2+2RB\cos(\phi'-\phi)}{R^2+B^2+2RB\cos(\phi'+\phi)} \ . \tag{20}$$

This infinite series without the factor $(-1)^{n+1}$ in its general term has its sum calculated in Ref. [1]: check eqs. (5) and (7). We can derive (20) by modifying the calculation performed there [notice the typo in eq. (6): it should be $\sigma(\theta'-\theta)-\sigma(\theta'+\theta)$] with only the addition of this factor. Setting $z \equiv (R/B)\,e^{i\varphi}$, we have

$$\sigma_-(\varphi) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\Big(\frac{R}{B}\Big)^n \cos n\varphi = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Re}\, z^n = \mathrm{Re}\int_0^z \Big[\sum_{n=0}^{\infty}(-\zeta)^n\Big]d\zeta$$

$$= \mathrm{Re}\int_0^z \frac{d\zeta}{1+\zeta} = \mathrm{Re}\ln(1+z) = -\ln|1+z| = \frac{1}{2}\ln\big[R^2+B^2+2BR\cos\varphi\big] - \ln B \ .$$

$$\therefore\ S_-(R,B) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big(\frac{R}{B}\Big)^n\big[\cos n(\phi'-\phi)-\cos n(\phi'+\phi)\big] = \sigma_-(\phi'-\phi)-\sigma_-(\phi'+\phi)$$

$$= \frac{1}{2}\ln\big[R^2+B^2+2BR\cos(\phi'-\phi)\big] - \frac{1}{2}\ln\big[R^2+B^2+2BR\cos(\phi'+\phi)\big] \ \checkmark$$

Looking at (19), we see that we have to use (20) with the parameters ϕ' and ϕ respectively equal to $\theta_<$ and $\pi-\theta_>$, in which case $\cos(\phi'\pm\phi) = -\cos(\theta'\mp\theta)$. Also noticing the property $S_-(R,B) = S_-(B,R)$, we can finally express G in exactly the same form as (2):

$$G = \frac{2S_-(r,r')+2S_-(rr',b^2)}{2} = \frac{1}{2}\ln\frac{r^2+r'^2-2rr'\cos(\theta'+\theta)}{r^2+r'^2-2rr'\cos(\theta'-\theta)} + \frac{1}{2}\ln\frac{r^2r'^2+b^4-2b^2rr'\cos(\theta'+\theta)}{r^2r'^2+b^4-2b^2rr'\cos(\theta'-\theta)} \ .$$

4 Final Comments

We used Green's formula in (7) with the factor $(2\pi)^{-1}$ so that our calculations would lead to exactly the same Green's function expression in (2) (deduced in Ref. [1]). Some authors, however, prefer this factor showing up in Green's function (and not in Green's formula); for example, compare eqs. (1.15b), (1.16), (1.17), and (1.22) in ch. 4 of Ref. [5] with eqs. (17) and (18) in Ref. [1].

The method can be applied in the case of other boundary conditions. Furthermore, it is not restricted to a semi-disk: the domain can be a generic circular sector (in which case the application of the method of images would no longer be simple).

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