Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

Chaos and Non-trivial Minimal Sets in Discontinuous Vector Fields

Claudio A. Buzzi, **Rodrigo D. Euzébio**,
Depto de Matemática, IBILCE, UNESP
15054-000, São José do Rio Preto, SP
E-mail: buzzi@ibilce.unesp.br, rodrigo.euzebio@sjrp.unesp.br

Tiago de Carvalho
Departamento de Matemática, FC, UNESP
17033–360, Bauru, SP
E-mail: tcarvalho@fc.unesp.br

Abstract: *In this work some aspects on chaotic behavior and minimality in planar discontinuous vector fields are treated. The occurrence of non-deterministic chaos is observed and the concept of minimality and orientable minimality are introduced. We also investigated some relations between minimality and orientable minimality and observed the existence of new kinds of non-trivial minimal sets in chaotic systems. The approach is geometrical and involve the ordinary techniques of the discontinuous dynamical systems theory.*

Keywords: *Discontinuous Vector Fields, Non-deterministic Chaos, Minimal Sets*

1 Setting the problem

Discontinuous vector fields (DVFs, for short) have become certainly one of the common frontiers between Mathematics and Physics or Engineering. Many authors have contributed to the study of DVFs (see for instance the pioneering work [4] or the didactic works [1, 7], and references therein about details of these multi-valued vector fields). In our approach Filippov's convention is considered. So, the vector field of the model is discontinuous across a *switching manifold* and it is possible for its trajectories to be confined onto the switching manifold itself. The occurrence of such behavior, known as sliding motion, has been reported in a wide range of applications. We can find important examples in electrical circuits having switches, in mechanical devices in which components collide into each other, in problems with friction, sliding or squealing, among others.

For planar smooth vector fields there is a very developed theory nowadays, mainly in the planar case. In such environment, questions about chaotic behaviour and minimality, for instance, are complety answered. Indeed, the Jordan curve theorem assures that there is no chaotic behaviour in planar systems and the Poincaré-Bendixson theorem says that for a given flow the minimal sets are just equilibria or limit cycles. Nevertheless, in higher dimension, while minimal sets are described by the Denjoy-Schwartz theorem (under some suitable hypothesis – see [6]), chaotic systems are massively studied and a final theory is far away from be reached. Nevertheless, a very interesting and useful subject is to study these kind of objects in the DVF's scenario. Furthermore, we must observe that chaotic behaviour and non-trivial minimality have been understudied in the DVF's literature.

The specific topic addressed in this work concerns with the occurrence of chaos in planar DVFs and the existence of non-trivial minimal sets in DVFs. As long as the authors know, a first study about the minimal set theory for DVFs and a discussion about the validity of the

Poincaré-Bendixson theorem for DVFs is given only in their paper [2]. Following the approach in [2], here we present some special DVFs and prove the existence of compact invariant sets with chaotic flow. Actually, these sets will be non-trivial minimal sets having no symmetry. We also propose definitions of minimal sets for positive (and negative) flow of DVFs (or orientable minimality) and study some relations between them and the definition of minimal set stablished in [2]. With these new definitions we analyze the occurrence of new kind of non-trivial minimal sets for DVFs defined in $\mathbb{R}^2$.

2 Preliminary definitions and support results

Let V be an arbitrarily small neighborhood of $0 \in \mathbb{R}^2$ and consider a codimension one manifold Σ of $\mathbb{R}^2$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$. Observe that we can assume, locally around the origin of $\mathbb{R}^2$, that $f(x, y) = y$.

Designate by χ the space of C^r-vector fields on $V \subset \mathbb{R}^2$, with $r \geq 1$ large enough for our purposes. Call Ω the space of vector fields $Z : V \to \mathbb{R}^2$ such that

$$Z(x,y) = \begin{cases} X(x,y), & \text{for} \quad (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for} \quad (x,y) \in \Sigma^-, \end{cases} \tag{1}$$

where $X = (X_1, X_2), Y = (Y_1, Y_2) \in \chi$. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept it to be multi-valued at points of Σ. The basic results of differential equations in this context were stated by Filippov in [4], that we summarize next. Indeed, consider Lie derivatives

$$X.f(p) = \langle \nabla f(p), X(p) \rangle \qquad \text{and} \qquad X^i.f(p) = \left\langle \nabla X^{i-1}.f(p), X(p) \right\rangle, i \geq 2$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$.

We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c .

(ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e.

(iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s.

In what follows we present the definition of local and global trajectories for DVFs. Before that, we remark that a tangency point of system (1) is characterized by $(X.f(q))(Y.f(q)) = 0$. If there exist a characteristic orbit of the vectors fields X or Y reaching q in a finite time, then such tangency is called a *visible tangency*. Otherwise we call q an *invisible tangency*. In addition, a tangency point p is *singular* if p is a invisible tangency for both X and Y. On the other hand, a tangential singularity p is *regular* if it is not singular.

The definition of local trajectory can also be found in [5].

Definition 1. *The* ***local trajectory (orbit)*** $\phi_Z(t,p)$ *of a DVF given by* (1) *is defined as follows:*

- *For* $p \in \Sigma^+ \backslash \Sigma$ *and* $p \in \Sigma^- \backslash \Sigma$ *the trajectory is given by* $\phi_Z(t,p) = \phi_X(t,p)$ *and* $\phi_Z(t,p) = \phi_Y(t,p)$ *respectively, where* $t \in I$.

- *For* $p \in \Sigma^c$ *such that* $X.f(p) > 0$, $Y.f(p) > 0$ *and taking the origin of time at* p, *the trajectory is defined as* $\phi_Z(t,p) = \phi_Y(t,p)$ *for* $t \in I \cap \{t \leq 0\}$ *and* $\phi_Z(t,p) = \phi_X(t,p)$ *for* $t \in I \cap \{t \geq 0\}$. *For the case* $X.f(p) < 0$ *and* $Y.f(p) < 0$ *the definition is the same reversing time.*

- *For* $p \in \Sigma^e$ *and taking the origin of time at* p, *the trajectory is defined as* $\phi_Z(t,p) = \phi_{Z^\Sigma}(t,p)$ *for* $t \in I \cap \{t \leq 0\}$ *and* $\phi_Z(t,p)$ *is either* $\phi_X(t,p)$ *or* $\phi_Y(t,p)$ *or* $\phi_{Z^\Sigma}(t,p)$ *for* $t \in I \cap \{t \geq 0\}$. *For the case* $p \in \Sigma^s$ *the definition is the same reversing time.*

- *For p a regular tangency point and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_1(t,p)$ for $t \in I \cap \{t \le 0\}$ and $\phi_Z(t,p) = \phi_2(t,p)$ for $t \in I \cap \{t \ge 0\}$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^Σ}.*

- *For p a singular tangency point $\phi_Z(t,p) = p$ for all $t \in \mathbb{R}$.*

The next definitions was stated in [2].

Definition 2. *A **global trajectory (orbit)** $\Gamma_Z(t,p_0)$ of $Z \in \chi$ passing through p_0 is a union*

$$\Gamma_Z(t,p_0) = \bigcup_{i \in \mathbb{Z}} \{\sigma_i(t,p_i); t_i \le t \le t_{i+1}\}$$

*of preserving-orientation local trajectories $\sigma_i(t,p_i)$ satisfying $\sigma_i(t_{i+1},p_i) = \sigma_{i+1}(t_{i+1},p_{i+1}) = p_{i+1}$ and $t_i \to \pm\infty$ as $i \to \pm\infty$. A global trajectory is a **positive** (respectively, **negative**) **global trajectory** if $i \in \mathbb{N}$ (respectively, $-i \in \mathbb{N}$) and $t_0 = 0$.*

In what follows we present the definitions of invariance and minimality.

Definition 3. *A set $A \subset \mathbb{R}^2$ is **Z-invariant** (respectively, **Z-positively/negatively invariant**) if for each $p \in A$ and all global trajectory $\Gamma_Z(t,p)$ (respectively, positive/negatively global trajectory $\Gamma_Z^{\pm}(t,p)$) passing through p it holds $\Gamma_Z(t,p) \subset A$ (respectively, $\Gamma_Z^{\pm}(t,p) \subset A$).*

Remark 1. *It follows directly from Definiton 3 that a given set is invariant if and only if it is Z-positively and Z-negatively invariant.*

Definition 4. *Consider $\mathbf{Z} \in \mathbf{\Omega}$. A set $M \subset \mathbb{R}^2$ is Z-**minimal** (respectively, Z-**positively/negatively minimal**) if*

(i) $M \neq \emptyset$;

(ii) M is compact;

(iii) M is Z-invariant (respectively, Z-positively/negatively invariant);

(iv) M does not contain proper subset satisfying (i), (ii) and (iii).

The following lemma is a trivial consequence of Definition 4.

Lemma 1. *Consider $M \in \mathbb{R}^2$ and Z a DVF. If M is Z-positively minimal and Z-negatively minimal, then M is Z-minimal.*

Proof. In fact, since M is Z-positively minimal and Z-negatively minimal, then M is a non-empty compact set and from Remark 1 M is Z-invariant and does not contain a proper non-empty compact Z-invariant subset. □

In what follows we introduce some definitions concerning objects from the ergodic theory into de context of DVFs.

Definition 5. *System* (1) *is topologically transitive on an invariant set W if for every pair of nonempty, open sets U and V in W, there exist $q \in U$, $\Gamma_Z^+(t,q)$ a positive global trajectory and $t_0 > 0$ such that $\Gamma_Z^+(t_0,q) \in V$.*

Definition 6. *System* (1) *exhibits sensitive dependence on a compact invariant set W if there is a fixed $r > 0$ satisfying $r < diam(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist a $y \in B_\varepsilon(x) \cap W$ and positive global trajectories Γ_x^+ and Γ_y^+ passing through x and y, respectively, satisfying*

$$d_H(\Gamma_x^+, \Gamma_y^+) = \sup_{a \in \Gamma_x^+, b \in \Gamma_y^+} d(a,b) > r,$$

where $diam(W)$ is the diameter of W and d is the Euclidean distance.

DOI: 10.5540/03.2015.003.01.0019

As observed in [3], the two previous definitions coincide for single-valued flows, making this a natural extension for a set-valued flow. Now we define a non-deterministic chaotic set:

Definition 7. *System* (1) *is chaotic on a compact invariant set* W *if it is topologically transitive and exhibits sensitive dependence on* W.

3 Results

Finding limit sets of trajectories of vector fields is one of the most important tasks of the qualitative theory of dynamical systems. In the literature there are several recent papers where the authors explicitly exhibit the phase portraits of some DVFs with their unfoldings. However, all the limit sets exhibited have trivial minimal sets (i.e., the minimal sets are equilibria, pseudo equilibria, cycles or pseudo-cycles). At this section we present some examples of non-trivial minimal set and chaotic DVFs.

Example 1. *Consider* $Z = (X, Y) \in \Omega$, *where* $X(x, y) = (1, -2x)$, $Y(x, y) = (-2, 4x^3 - 2x)$ *and* $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$. *The parametric equation for the integral curves of* X *and* Y *with initial conditions* $(x(0), y(0)) = (0, k_+)$ *and* $(x(0), y(0)) = (0, k_-)$, *respectively, are known and its algebraic expressions are given by* $y = -x^2 + k_+$ *and* $y = x^4/2 - x^2/2 + k_-$, *respectively. It is easy to see that* $p = (0, 0)$ *is an invisible tangency point of* X *and a visible one of* Y. *It is also easy to note that the points* $p_\pm = (\pm\sqrt{2}/2, 0)$ *are both invisible tangency points of* Y. *Note that between* p_- *and* p *there exists an escaping region and between* p *and* p_+ *a sliding one. Further, every point between* $(-1, 0)$ *and* p_- *or between* p_+ *and* $(1, 0)$ *belong to a sewing region. Consider now the particular trajectories of* X *and* Y *for the cases when* $k_+ = 1$ *and* $k_- = 0$, *respectively. These particular curves delimit a bounded region of plane that we call* Λ *and it is the main object of this section. Figure 1 summarizes these facts.*

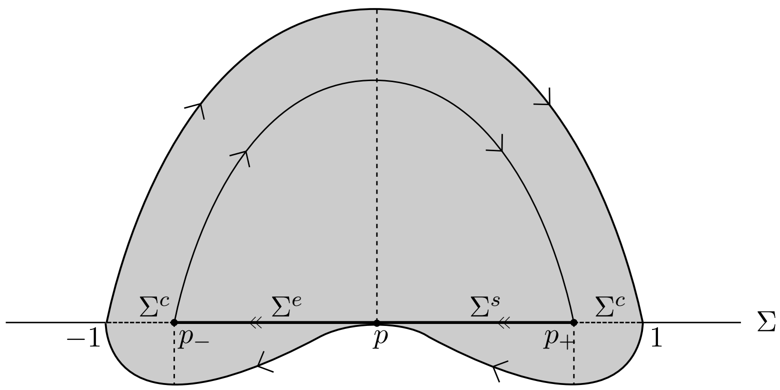

Figura 1: Special integral curves and tangency points.

Proposition 1. *Consider* $Z = (X, Y) \in \Omega$, *where* $X(x, y) = (1, -2x)$, $Y(x, y) = (-2, 4x^3 - 2x)$ *and* $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$. *The set*

$$\Lambda = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1 \text{ and } x^4/2 - x^2/2 \leq y \leq 1 - x^2\}. \tag{2}$$

is a minimal set for Z.

Demonstração. It is easy to see that Λ is compact and has non-empty interior. Moreover, by Definition 1, on $\partial\Lambda \setminus \{p\}$ we have uniqueness of trajectory (here ∂B means the boundary of the set B). Note that a global trajectory of any point in Λ meets p for some time t^*. Since p is a visible tangency point for Y and $p \in \overline{\partial\Sigma^e} \cap \overline{\partial\Sigma^s}$, according to the fourth bullet of Definition 1 any trajectory passing through p remain in Λ. Consequently Λ is Z-invariant. Moreover, given

DOI: 10.5540/03.2015.003.01.0019

$p_1, p_2 \in \Lambda$ the positive global trajectory by p_1 reaches the sliding region between p and p_+ and slides to p. The negative global trajectory by p_2 reaches the escaping region between p and p_- and slides to p. So, there exists a global trajectory connecting p_1 and p_2. Now, let $\Lambda' \subset \Lambda$ be a Z-invariant set. Given $q_1 \in \Lambda'$ and $q_2 \in \Lambda$ since there exists a global trajectory connecting them we conclude that $q_2 \in \Lambda'$. Therefore, $\Lambda' = \Lambda$ and Λ is a minimal set. □

Example 2. *The set* Λ_1 *(see Figure 2)defined by some particulars integral curves of the vector field* $Z_1 = (X, Y) \in \Omega$, *defined by* $X(x, y) = (1, -2x+1)$, $Y(x, y) = (-1, (-2+x)(-22+x(-7+4x)))$ *with* $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$ *is* Z_1*-minimal but it is neither* Z_1*-positively minimal nor* Z_1*-negatively minimal.*

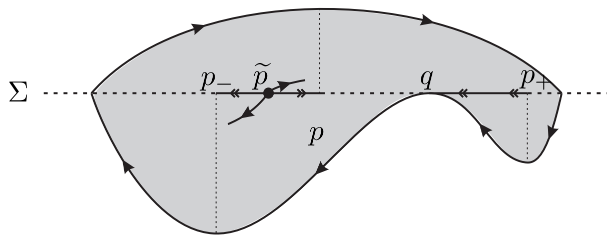

Figura 2: Z_1-minimal set Λ_1. Λ_1 is neither Z_1-positively minimal nor Z_1-negatively minimal.

Example 3. *The* Z*-minimal set presented in Proposition 1 is also* Z*-positively minimal and* Z*-negatively minimal. The proof of this fact follows the same lines of the proof of Proposition 1.*

Example 4. *Consider* Z_2 *a DVF presenting the phase portrait exhibited in Figure 3 with* $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$ *and* Λ_2 *the set indicated in gray. Then the set* Λ_2 *is* Z_2*-minimal and also* Z_2*-positively minimal but not* Z_2*-negatively minimal for this DVF.*

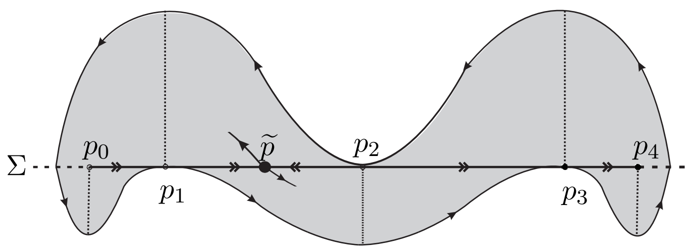

Figura 3: The Z_2-minimal set Λ_2.

Next theorem says that the vector field presented in Proposition 1 is chaotic under Λ.

Theorem 8. *Consider* $Z = (X, Y) \in \Omega$ *and* Λ *as in Proposition 1, where* $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$*. Then the planar DVF* Z *is chaotic on* Λ*.*

Before proving Theorem 8 we present the following lemma. It will be fundamental in the proof of Theorem 8.

Lemma 2. *Consider the set* Λ *defined in Theorem 8. Then, for any* $x, y \in \Lambda$*, there exist a positive global trajectory* $\Gamma^+(t, x)$ *passing through* x *and* $t_0 > 0$ *such that* $\Gamma^+(t_0, x) = y$*.*

The previous lemma says that any two points in Λ can be connected by some positive global trajectory. Its proof is straighfoward if we observe that a global trajectory of any point in Λ meets p for some time t^*, as the authors argued in [2]. Now we prove Theorem 8.

Proof of Theorem 8. In order to prove that the DVF Z is topologically transitive on Λ, we observe that Λ is compact and invariant since it is minimal (see Proposition 1 of [2]). Now consider nonempty open sets U and V in Λ. Since U and V are nonempty, there exist at least an element λ_1 in U and another one λ_2 in V. By Lemma 2, there exist a positive global trajectory $\Gamma^+(t, \lambda_1)$ passing through λ_1 and $t_0 > 0$ such that $\Gamma^+(t_0, \lambda_1) = \lambda_2 \in V$. Consequently the DVF Z is topologically transitive on the invariant set Λ.

Now we prove that Z exhibits sensitive dependence on Λ. Indeed, take $m = diam(\Lambda)$ and consider $r = m/2 > 0$. Since $r < m$ then there exists two elements a and b in Λ such that $d(a, b) > r$. Now consider $x \in \Lambda$, $\varepsilon > 0$ and fix $y \in B_\varepsilon(x) \cap \Lambda$. By Lemma 2 there exist positive global trajectories $\Gamma^+_Z(t, x)$ of x and $\Gamma^+_Z(t, y)$ of y and numbers $t_1, t_2 > 0$ such that $\Gamma^+_Z(t_1, x) = a$ and $\Gamma^+_Z(t_2, y) = b$. Then $d_H(\Gamma^+_Z(t_1, x), \Gamma^+_Z(t_2, y)) = d(a, b) > r$ and consequently Z exhibits sensitive dependence on Λ. Thus the planar DVF Z is chaotic on the invariant compact set Λ. □

The following result indicates the presence of chaos in the DVF Z_2 studied in this section.

Theorem 9. *Consider the DVF Z_2 and the set Λ_2 as presented in Example 4. Then Z_2 is chaotic on Λ_2.*

Proof. The proof of Theorem 9 follows the same lines of the proof of Theorem 8 by using a similar result to Lemma 2 for the Z-minimal set Λ_2. □

One should note that Theorems 8 and 9 present examples of PSVFs that are chaotic on minimal sets. This fact suggests a relation between chaoticity and minimality in PSVF that we make clear in the following theorem.

In what follows we denote by $med(\cdot)$ the Lebesgue measure.

Theorem 10. *Let Z be a planar DVF and $\Lambda \subset \mathbb{R}^2$ a compact invariant set. If Λ is Z-positively minimal and Z-negatively minimal satisfying $med(\Lambda) > 0$, then Z is chaotic on Λ.*

Theorem 10 is a very interesting result because presents a connection between two important different objects of the recent theory of DVF, namely, the chaotic planar systems and the non-trivial Z-minimal sets.

In order to prove Theorem 10, we introduce the next two lemmas. The first one is a generalization of Lemma 2.

Lemma 3. *Under the same hypotheses of Theorem 10, it holds that for any $x, y \in \Lambda$, there exist a global trajectory $\Gamma(t, y)$ passing through y and $t^* > 0$ such that $\Gamma^+(t^*, y) = x$.*

Proof. Since $med(\Lambda) > 0$, by Poincaré-Bendixson Theorem for DVF presented in [2], there exist at least a set $A \subset \Sigma \cap (\Sigma^e \cup \Sigma^s)$. Otherwise, we have $\Sigma \cap \Lambda = \Sigma^c \cup \Sigma^t$ and then by the referred theorem we get $med(\Lambda) = 0$, where Σ^t is the set of tangencies points of Z. For each $a \in A$, denote by Π^+_a the set of all positive global trajectories passing through a and by Π^-_a its negative analogous. Now consider the sets

$$A^\pm_a = \bigcup_{\Gamma_a \in \Pi^\pm_a} \Gamma_a(t, a) \subset \Lambda.$$

Actually we have $A^\pm_a = \Lambda$, since $A^\pm_a$ is Z-positively (respectively negatively) invariant restrained in the Z-positively (respectively negatively) minimal set Λ. In order to see that A^+_a is Z-positively invariant, let p be a point in A^+_a and $\Gamma_p(t, p)$ a positive global trajectory passing through p. Since $p \in A^+_a$, then there exists a positive global trajectory $\tilde{\Gamma}_a(t, a)$ passing through a and $t_0 > 0$ such that $\tilde{\Gamma}_a(t_0, a) = p$. Consequently $\Gamma_p(t, p)$ belongs to A^+_a once it is restrained to the positive global trajetory $\hat{\Gamma}_a(t, a) = \tilde{\Gamma}_a(t, a) \cup \Gamma_p(t, p) \subset A^+_a$. Analogously we can prove that A^-_a is Z-negatively invariant.

Now consider $x, y \in \Lambda$ arbitrary points. Since $A_a^- = \Lambda = A_a^+$, there exists $\Gamma_a^+(t,a) \in A_a^+$ a positive global trajectory, $\Gamma_a^-(t,a) \in A_a^-$ a negative global trajectory and values $t_x > 0$, $t_y < 0$ such that $\Gamma_a^+(t_x,a) = x$ and $\Gamma_a^+(t_y,a) = y$. Consequently there exists a global trajectory $\Gamma(t,y)$ passing through y and $t^* = t_x + |t_y| > 0$ such that $\Gamma(t^*,y) = x$. □

Lemma 4. *Under the same hypotheses of Theorem 10, if any two points of* Λ *can be connected by a global trajectory of* Z*, then* Z *is chaotic on* Λ*.*

Proof. The proof of Lemma 4 is similar to the proof of Theorem 8 by using Lemma 3 instead of Lemma 2. □

Proof of Theorem 10. The proof is straightforward from Lemmas 3 and 4. □

We stress that in each example through this work we have $\Sigma^e \cap \Sigma^s \neq \emptyset$. Nevertheless, non-trivial minimal sets can not happen when $\Sigma^e \cap \Sigma^s = \emptyset$, as proved in [2]. Indeed, in such work the authors introduce a version of the classical Poincaré-Bendixson Theorem for DVFs without sliding motion. There it is shown that the limit sets are the classical ones besides pseudo-cycles, pseudo-graphs and singular tangencial singularities.

4 Conclusions

In this work we have introduced definitions of Z-minimal sets of DVFs taking into account the fact that DVFs have a strong dependence of the orientation of the trajectories, as we can see in Definition 1. Moreover, we verified the existence of non deterministic chaos in planar DVFs whithout symmetry or coincidence of invisible tangencies (Canard phenomena). As far as the authors know, this is the first time that non-smooth systems with such characteristic are observed in the planar case. Finally, we verify the presence of chaotic behavior in planar DVFs and present a result relating chaotic behaviour with orientable minimality, which emphasizes the importance of providing the definition of orientable minimality.

Referências

[1] M. di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise-smooth Dynamical Systems – Theory and Applications*, Springer-Verlag (2008).

[2] C.A. Buzzi, T. de Carvalho and R.D. Euzebio, *On Poincare-Bendixson Theorem and non-trivial minimal sets in planar nonsmooth vector fields*, submitted (arxiv link: http://arxiv.org/pdf/1307.6825v1.pdf).

[3] A. Colombo and M.R. Jeffrey, *Nondeterministic chaos, and the two-fold singularity in piecewise smooth flows*, SIAM J. Applied Dymanical Systems, 10**2**, (2011), 423–451.

[4] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers-Dordrecht, 1988.

[5] M. Guardia, T.M. Seara and M.A. Teixeira, *Generic bifurcations of low codimension of planar Filippov Systems*, Journal of Differential Equations **250** (2011) 1967–2023.

[6] C. Gutierrez, *Smoothing continuous flows and the converse of Denjoy-Schwartz Theorem*, Anais Academia Brasileira de Ciências, 51(4) (1979), 581–589.

[7] M.A. Teixeira, *Perturbation Theory for Non-smooth Systems*, Meyers: Encyclopedia of Complexity and Systems Science **152** (2008).

DOI: 10.5540/03.2015.003.01.0019