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Optimal Control of Fractional Punishment in Optional Public Goods Game

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Improving cooperation is a key issue in many systems and organizations. Punishment is a mechanism to improve cooperation, but it can be expensive [1], and consequently it itself becomes a Public Good [2]. Several mechanism to implement punishment are encountered in the literature including sanctioning only a fraction of the free riders [3]. This approach reduces the number of free riders, and consequently the cost of the sanctioning system.

In this work, we seek to improve the cooperation and optimize the fractional punishment in Optional Public Goods Games (OPGG) [4] by formulating a minimization with restriction problem. To this end, let (t_0, t_f) denote a time interval, $w(t)$ the state variable and $\mathcal{W} \subset \mathbb{R}^3$ the state space. We consider $\mathcal{W} = \mathcal{S}_3$ defined as $\mathcal{S}_3 = \left\{[x, y, z]^T \in \mathbb{R}^3 : x, y, z \geq 0 \text{ and } x + y + z = 1\right\}$. Given an initial condition $w(0) \in \mathcal{W}$ the state equation has the form [3]:

$$\dot{x} = x\,(p_x(w) - \bar{p}(w, v)) \tag{1}$$
$$\dot{y} = y\,(p_y(w, v) - \bar{p}(w, v)) \tag{2}$$
$$\dot{z} = z\,(p_z - \bar{p}(w, v)) \tag{3}$$

where $w(t) = [x(t), y(t), z(t)]^T$ such that each entry $0 \leq x(t), y(t), z(t) \leq 1$ is the frequency of each of the corresponding available strategies of the population at a specific time t (cooperators, defectors and loners, respectively). Each p_i is the payoff of the ith strategy, and $\bar{p}$ is its average given by $\bar{p} = xp_x + yp_y + zp_z$.

The distributed control is denoted by $v \in [0, 1] = \mathcal{U}$. We indicate the dependence of w on $v \in \mathcal{U}$ using the notation $w(v)$. Given a target function w^* in $L^2(t_0, t_f)$ and parameters $\alpha_i > 0$, the cost function is [5]:

$$\begin{aligned} J(w(v), v) \;=\; & \frac{\alpha_1}{2}\|w(t_f, v) - w^*\|_2^2 + \frac{\alpha_2}{2}\int_{t_0}^{t_f} \|w(\tau, v) - w^*\|_2^2 d\tau \\ & + \frac{\alpha_3}{2}\int_{t_0}^{t_f} \|v(\tau)\|_2^2 d\tau + \frac{\alpha_4}{2}\int_{t_0}^{t_f} \|v(\tau)y(\tau)\|_2^2 d\tau \end{aligned} \tag{4}$$

Preliminary results are obtained with $\alpha_1 = 0.01$, $\alpha_2 = 0.8$, $\alpha_3 = 0.0001$, $\alpha_4 = 20$ and $w^* = [1, 0, 0]^T$. The optimized trajectory is contrasted with two others, obtained with two values of v that generate qualitatively different results. Trajectory results are presented in Figure 1(a) while the control efforts are presented in Figure 1(b). It can be observed that the controlled trajectory avoids cycling and getting as close as small control efforts do to the full loner undesired state $[0, 0, 1]^T$ while at the same time maintaining a lower value than a stronger constant control would in later time values. Considering the cost of punishment at an instant $C = kNy(t)v(t)$, noting that k and N are constants related to the cost of punishing a single individual and the number of

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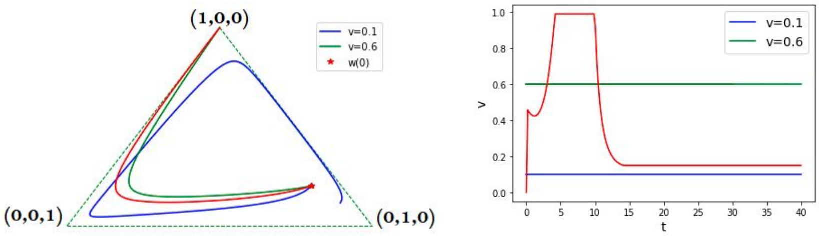

(*a*) (*b*)

Figure 1: (a)Trajectory results of simulations in the state space. Red trajectory corresponds to variable control effort. (b) Control efforts $v(t)$ over time.

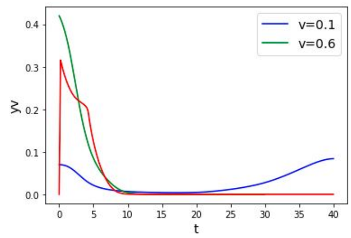

Figure 2: Magnitude of variable yv over simulation time t.

people in the population respectively, the magnitude of $y(t)v(t)$ presented in Figure 2 gives us an insight into what strategy is more cost effective in financial terms. Comparing the red and green curves in Figure 2 is observed that the red curve has a smaller integration value, corresponding to a lower overall cost. The blue curve is not taken into consideration because of the cycles.

The results evidence that the fractional punishment of OPGG can be efficiently controlled to improve the cooperation while at the same time reducing the cost of the sanctioning system.

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