Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 9, n. 1, 2022.

Trabalho apresentado no XLI CNMAC, Unicamp - Campinas - SP, 2022.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics
Preprint

Statistical Properties for Trigonometric Random Fields

Eduardo S. Schneider[1]
Universidade Federal de Pelotas, Pelotas, RS, Brazil

Abstract. This work presents a general form for a scalar random field which is written as a sum of finitely many Fourier modes. We get some of its statistical proprieties and analyze its geometry. Additionally, we derive a model for a Gaussian, two-dimensional, mean-zero, homogeneous, steady, and incompressible random velocity field and provide numerical evidence about the non-normality of the joint distribution of the Lagrangian velocity process.

Keys words. Random velocity fields, Gaussian, Fourier modes, passive trace transport.

1 Introduction

The problem of obtaining the statistical descriptions of the motion of one single particle in a random velocity field has being studied for many years. In particular, the passive tracer transport problem, which consists of determining the probability law of the position $\mathbf{X}_t$, for $t \geq 0$, of one single particle at time $t \geq 0$, which is moved by a random velocity field $\mathbf{U}$ when the motion of the particle does not affect the random velocity field. For references, see [4, 5, 9, 10]. In Figure 1, we present a single realization of a velocity field $\mathbf{U}$ and the trajectory of one particle starting its motion at $\mathbf{X}_0 = 0$ from time $t = t_0$ to time $t = t_1$, which is a useful visual tool to understand and exemplify some key concepts.

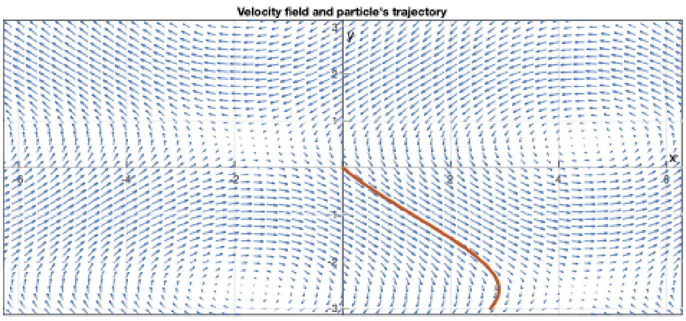

Figure 1: Velocity field and trajectory of one single particle from $t_0 = 0$ to $t = t_1$.

Let $\mathbf{U} = \{\mathbf{U}(\mathbf{x}, t), \mathbf{x} \in \mathbb{R}^2, t \geq 0\}$ be a random velocity field and let $\mathbf{X}_t$ be the particle position at time t, for $t \geq 0$. So $\mathbf{X}_t, t \geq 0$ is the solution of the differential equation of the motion given by

$$\frac{d\mathbf{X}_t}{dt} = \mathbf{U}(\mathbf{X}_t, t), \quad t > 0; \quad \mathbf{X}_0 = \mathbf{0}. \tag{1}$$

We are interested in determining the law of the entire stochastic location process $\mathbf{X} = \{\mathbf{X}_t, t \geq 0\}$, given the law of the random velocity field $\mathbf{U}$. However, despite much work being done on this

[1]eduardo.schneider@ufpel.edu.br

problem, we still have only a limited ability to derive results about the law of the particle position $\mathbf{X}_t$ from the law of the velocity field $\mathbf{U}(\mathbf{x}, t)$.

A related and important problem is to determine the law of the Lagrangian velocity process $\mathbf{U} = \{\mathbf{U}(\mathbf{X}_t, t), t \geq 0\}$, which is the particle's velocity viewed by an observer whose location $\mathbf{X}_t$ is determined by the environment. Different from the Eulerian description provided by $\mathbf{U}(\mathbf{x}, t)$, for which the coordinate system is fixed, the Lagrangian description gives us a description of the velocity field from the view of a particle following the velocity field.

Remark 1.1. *In a previous work [8], we use trigonometric velocity fields, as in Eq(6), to get the first terms of the Taylor expansion for the Lagrangian auto-covariance function, which is a piece of important statistical information for the Lagrangian velocity process* $\mathbf{U}$.

For this work we assume a scalar random field $S(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^2$, for $t \geq 0$, also called a stream function, written as a sum of finitely many Fourier modes. Then we present some theoretical results for its statistical properties and prescribe conditions for which every sum of finitely many Fourier modes provides a scalar Gaussian random field. Additionally, we analyze its geometry.

By considering the perpendicular gradient of the stream function $S(\mathbf{x}, t)$ we present a model for an incompressible random velocity field $\mathbf{U}(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^2$, $t \geq 0$ which is also written as a sum of finitely many Fourier modes and has similar statistical properties as the stream function $S(\mathbf{x}, t)$. In addition, we explore numerical simulations for particle trajectories and the Lagrangian velocity $\mathbf{U}(\mathbf{X}_t, t)$, for $t \geq 0$, for steady Gaussian trigonometric velocity fields. In fact, we are able to present numerical evidence that the joint distribution of $(\mathbf{U}(\mathbf{0}), \mathbf{U}(\mathbf{X}_t))$, for each $t > 0$, is not Gaussian.

2 Trigonometric Stream Functions

Let us start by setting a stream function written as a sum of finitely many Fourier modes as

$$S(\mathbf{x}) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} R_n \cos(\mathbf{W}_n \cdot \mathbf{x} + \Phi_n), \quad \mathbf{x} \in \mathbb{R}^2, \tag{2}$$

where the random amplitudes R_n and random wave numbers $\mathbf{W}_n$ are independent of the random phases Φ_n, in the sense that the collection $(R_1, \mathbf{W}_1, R_2, \mathbf{W}_2, \cdots, R_N, \mathbf{W}_N)$ is independent of $(\Phi_1, \Phi_2, \ldots, \Phi_N)$. Furthermore, we assume that the random phases Φ_n, $n = 1, 2, \ldots, N$, are independent and uniformly distributed on $[0, 2\pi]$, and the random vectors $(R_n, \mathbf{W}_n)$, $n = 1, 2, \ldots, N$, are square-integrable.

2.1 Some Statistical Proprieties for $S(\mathbf{x})$

The stream function $S(\mathbf{x})$ as defined above have some nice statistical properties.

Proposition 2.1. *Let* $S(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, *be a random field as in Eq.(2). Then* $S(\mathbf{x})$ *is mean-zero.*

Proof. For details, see [7]. □

Definition 2.1. *A random field* $S(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, *is said to be strictly homogeneous if, for all* $m \in \mathbb{N}$ *and* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbb{R}^2$, *the random vector* $(S(\mathbf{x}_1 + \mathbf{x}), S(\mathbf{x}_2 + \mathbf{x}), \ldots, S(\mathbf{x}_m + \mathbf{x}))$ *has the same distribution, for all* $\mathbf{x} \in \mathbb{R}^2$.

Theorem 2.1. *Let* $S(\mathbf{x})$ *be a random field as in Eq.(2). Then* $S(\mathbf{x})$ *is strictly homogeneous.*

Proof. For details, see [7]. □

DOI: 10.5540/03.2022.009.01.0255

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics. v. 9, n. 1, 2022.

Remark 2.1. *Notice that the stream function $S(\mathbf{x})$ is a mean-zero, strictly homogeneous, and stationary scalar random field. Later, we construct a bi-dimensional and incompressible random velocity field $\mathbf{U}(\mathbf{x}, t)$ that preserves these same statistical proprieties from $S(\mathbf{x})$.*

2.1.1 Gaussian Stream Functions

Assuming the general form for $S(\mathbf{x})$ as described in Eq.(2), we are able to prescribe conditions for which every sum of finitely many Fourier modes provides a scalar Gaussian random field.

Starting with the case $N = 1$, that is, the scalar random field $S(\mathbf{x})$ is written as

$$S(\mathbf{x}) = R\cos(\mathbf{W}\cdot\mathbf{x} + \Phi), \quad \mathbf{x} \in \mathbb{R}^2, \tag{3}$$

where $(R, \mathbf{W})$ is independent of Φ and Φ is uniformly distributed on $[0, 2\pi]$. We want to assign a distribution for random vector $(R, \mathbf{W})$ that makes $S(\mathbf{x})$ normally distributed, for every $\mathbf{x} \in \mathbb{R}^2$.

Remark 2.2. *According to Theorem 2.1, $S(\mathbf{x})$ is strictly homogeneous and then $S(\mathbf{0})$ and $S(\mathbf{x})$ have the same distribution, for all $\mathbf{x} \in \mathbb{R}^2$. Furthermore, we have that*

$$S(\mathbf{0}) = R\,\cos(\Phi), \tag{4}$$

and so the distribution of $S(\mathbf{0})$ depends only on the distributions of random variables R and Φ.

Remark 2.3. *Notice that by setting $Z_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ and $Z_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$, where U_1 and U_2 are independent and uniformly distributed on $[0, 1]$, we get a pair of independent and normally distributed random variables with mean 0 and variance 1, see [2]. Then we can set $R = \sqrt{-2\ln U_1}$, in this case, R has density $f_R(r) = \frac{1}{2}|r|e^{-\frac{1}{2}r^2}$, for $r \in \mathbb{R}$, and $\Phi = 2\pi U_2$, in Eq.(4), so that $S(\mathbf{0})$ is normally distributed.*

Lemma 2.1. *Let $S(\mathbf{x})$ be as in Eq.(3), where random variables $R = \sqrt{-2\ln U_1}$ and $\Phi = 2\pi U_2$ with U_1 and U_2 independent and uniformly distributed on $[0, 1]$, and $\mathbf{W} = \mathbf{w}$ with probability 1, for some fixed $\mathbf{w} \in \mathbb{R}^2$. Then, $S(\mathbf{x})$ is jointly Gaussian.*

Proof. For details, see [7]. □

Remark 2.4. *Assuming that random vectors $(R_n, \mathbf{W}_n, \Phi_n)$, for $n = 1, 2, \ldots, N$, are independent and identically distributed and $\mathbb{P}(\mathbf{W}_n = \mathbf{w}_n) = 1$, for n=1,2,..., N, for given $w_1, w_2, \ldots, w_N \in \mathbb{R}^2$. Notice that by extending the results on Remark 2.3 and on Lemma 2.1 we can get jointly Gaussian random fields, as in Eq.(2), with finitely many Fourier modes. In particular, $S(\mathbf{0})$ is Gaussian.*

2.2 The Geometry of the Stream Function

Notice that stream function $S(\mathbf{x})$, as in Eq.(2), is a function defined on $\mathbb{R}^2$ that assumes values on $\mathbb{R}$, so it is suitable and informative to represent $S(\mathbf{x})$ using contour plots. We want to analyze how the number of Fourier modes N changes the stream function $S(\mathbf{x})$ for different values of N.

Considering the case $N = 1$, we have

$$S(\mathbf{x}) = R\,\cos(\mathbf{W}\cdot x + \Phi), \quad \mathbf{x} \in \mathbb{R}^2, \tag{5}$$

where $R \in \mathbb{R}$, $\mathbf{W} \in \mathbb{R}^2$, and $\Phi \in [0, 2\pi]$. The contour plot associated to this stream function consists of parallel straight lines in the plane. The parameter $\mathbf{W}$ provides the slope of such straight lines; the parameter R gives the amplitude of $S(\mathbf{x})$; and the parameter Φ acts as a spatial translation. In Figure 2, we present one realization for a stream function $S(\mathbf{x})$ with only 1 Fourier mode.

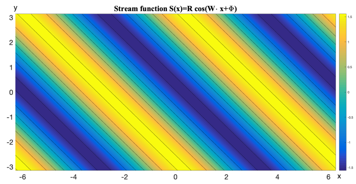

Figure 2: Stream function $S(x)$ for a model with 1 Fourier mode.

For the case $N = 2$, by setting $R_1 = R_2 = 1$ and choosing distinct combinations for $\mathbf{W}_1$ and $\mathbf{W}_2$ we notice different configurations for level curves of $S(\mathbf{x})$, see Figure 3. The visual representation, however, does not change for different values of R as long as they are constant and not 0. We present two extreme cases, when $\mathbf{W}_1$ and $\mathbf{W}_2$ are parallel and perpendicular, and two intermediate cases to illustrate how the form of level curves change according to wave numbers.

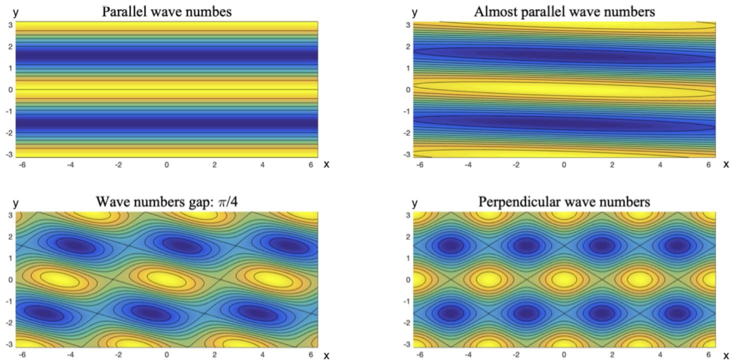

Figure 3: Stream function $S(\mathbf{x})$ for a model with 2 Fourier modes.

As we increase the number of Fourier modes of the stream function $S(\mathbf{x})$ then it becomes more complex, for example, it is harder to identify some periodicity on contour plots of $S(\mathbf{x})$. In Figure 4, we present contour plots for a model with random wave-numbers and distinct values of N.

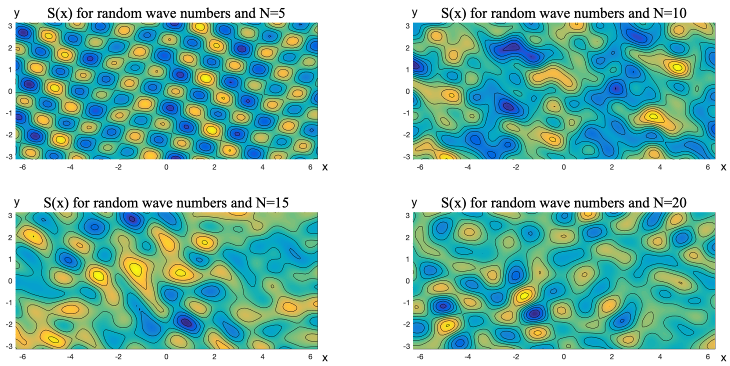

Figure 4: Stream functions $S(\mathbf{x})$ for $N = 5$, $N = 10$, $N = 15$, and $N = 20$ Fourier modes.

DOI: 10.5540/03.2022.009.01.0255

2.3 Trigonometric Velocity Fields

Starting with the stream function described in Eq.(2), we can define a divergence-free velocity filed $\mathbf{U}(\mathbf{x})$, for $\mathbf{x} \in \mathbb{R}^2$, by considering the perpendicular gradient of such stream function, that is,$\mathbf{U}(\mathbf{x}) = \nabla^{\perp} S(\mathbf{x})$, or, more explicitly, writing

$$\mathbf{U}(\mathbf{x}) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} R_n \sin(\mathbf{W}_n \cdot \mathbf{x} + \Phi_n)\, \mathbf{\Theta}_n, \quad x \in \mathbb{R}^2, \tag{6}$$

where $\mathbf{\Theta}_n = \mathbf{W}_n^{\perp}$ is a counter clockwise rotation of $\mathbf{W}_n$ by angle $\pi/2$, collections of random variables $(R_1, \mathbf{W}_1, R_2, \mathbf{W}_2, \cdots, R_N, \mathbf{W}_N)$ and $(\Phi_1, \Phi_2, \ldots, \Phi_N)$ are independent and random phases Φ_n, $n = 1, 2, \ldots, N$, are independent and uniformly distributed on $[0, 2\pi]$.

Remark 2.5. *Notice that $\mathbf{U}(\mathbf{x})$, as in Eq.(6), is a two-dimensional, mean-zero, strictly homogeneous, stationary, and incompressible random field. For details, see [7]. Such properties are key statistical properties of real turbulence. Unlike real turbulence, however, $\mathbf{U}(\mathbf{x})$ does not depend on t. Moreover, those statistical proprieties for a random field match with the properties of random fields described in many works available in the literature, for references see [1, 3, 4].*

Remark 2.6. *Consider the stream function $S(\mathbf{x})$ as in Eq(2), under the conditions prescribed by the Remark 2.4, and the velocity field $\mathbf{U}(\mathbf{x})$ as in Eq.(6). Assume that one single particle moves according to the random velocity field, satisfying the differential equation of the motion given by Eq.(1). So the Lagrangian velocity $\mathbf{U}(\mathbf{0})$, at $t = 0$, is Gaussian since each component of $\mathbf{U}(\mathbf{0})$ has the same distribution as $S(\mathbf{0})$.*

Remark 2.7. *According to [6, 11], the Lagrangian velocity process $\mathbf{U}(\mathbf{X}_t, t)$, for $t \geq 0$, is stationary and so random fields $\mathbf{U}(\mathbf{X}_0, 0)$ and $\mathbf{U}(\mathbf{X}_t, t)$ have the same distribution, for all $t \geq 0$. In particular, $\mathbf{U}(\mathbf{X}_0, 0)$ and $\mathbf{U}(\mathbf{X}_t, t)$ are Gaussian. However, we do have only limited information about the joint distribution of $\mathbf{U}(\mathbf{X}_0, 0)$ and $\mathbf{U}(\mathbf{X}_t, t)$, for $t > 0$.*

2.4 Numerical Simulations for the Lagrangian velocity

Let us consider a simple velocity field $\mathbf{U}(\mathbf{x})$ with only two Fourier modes and deterministic wave numbers $\mathbf{w}_1 = (1, 0)$ and $\mathbf{w}_2 = (0, 1)$, which are perpendicular and have the same modulus equals 1. Assuming that random vectors (R_1, Φ_1) and (R_2, Φ_2) are independent, random amplitudes R_1 and R_2 are independent and identically distributed with density $f_{R_i}(r) = \frac{1}{2}|r|e^{-\frac{1}{2}r^2}$, for $r \in \mathbb{R}$, for $i = 1, 2$, and random phases Φ_1 and Φ_2 are independent and uniformly distributed on $[0, 2\pi]$. So

$$\mathbf{U}(\mathbf{x}) = \frac{1}{\sqrt{2}} \left(R_1 \sin(\mathbf{w}_1 \cdot \mathbf{x} + \Phi_1)\theta_1 + R_2 \sin(\mathbf{w}_2 \cdot \mathbf{x} + \Phi_2)\theta_2 \right), \quad \mathbf{x} \in \mathbb{R}^2, \tag{7}$$

is is a mean-zero, strictly homogeneous, stationary, and incompressible random velocity field and $\mathbf{U}(\mathbf{0})$ is Gaussian. Moreover, according to Remark 2.7, the Lagrangian velocity $\mathbf{U}(\mathbf{X}_t)$, at time $t > 0$, has the same distribution as $\mathbf{U}(\mathbf{0})$.

Assume that one single particle moves according Eq.(1). We set an experiment that consists of repeatedly dropping a particle on velocity field $\mathbf{U}(\mathbf{x})$ at location $\mathbf{X}_0 = 0$, for $t_0 = 0$, for many realizations of the velocity field $\mathbf{U}(\mathbf{x})$, track it, and then analyze Lagrangian velocities $\mathbf{U}(\mathbf{0})$, at time $t_0 = 0$, and $\mathbf{U}(\mathbf{X}_{t_1})$, at time $t_1 = 5$. We numerically generate a large number of realizations of such experiment, then we use quantile-quantile plots, or just called Q-Q plots, which consist of a graphical method for comparing two probability distributions. In this case, we are looking to see whether the data fits or not a straight line. If it does, the Q-Q plot suggests that data came from a normal distribution, if it does not then data are not from a normal distribution. Thus, we analyze and compare data coming from $\mathbf{U}(\mathbf{0}, 0)$ and $\mathbf{U}(\mathbf{X}_t, t)$ with data from a Gaussian distribution.

Remark 2.8. *According to Figure 5, data from each component of* $\mathbf{U}(\mathbf{0})$ *fits almost perfectly to the straight lines. Similar Q-Q plots show the same visual representation for* $\mathbf{U}(\mathbf{X}_t)$*. Such representations suggest that each component of* $\mathbf{U}(\mathbf{0})$ *and* $\mathbf{U}(\mathbf{X}_t)$ *is Gaussian. Moreover, in Figure 6, data from linear combinations of components of* $\mathbf{U}(\mathbf{0})$ *or* $\mathbf{U}(\mathbf{X}_t)$ *also fits very well to the straight lines, suggesting that linear combinations of components of* $\mathbf{U}(\mathbf{0})$ *and* $\mathbf{U}(\mathbf{X}_t)$ *are also Gaussian. These facts agree with the results from Remarks 2.7 and Remark 2.6.*

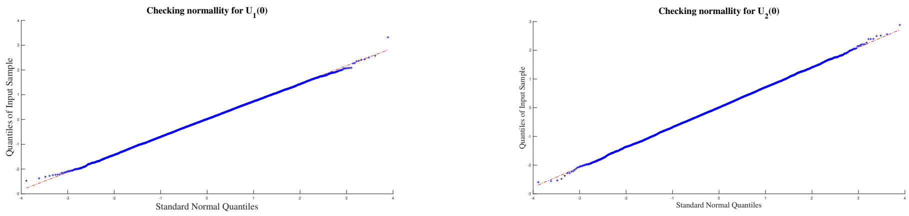

Figure 5: Q-Q plots for each component of $\mathbf{U}(\mathbf{0})$.

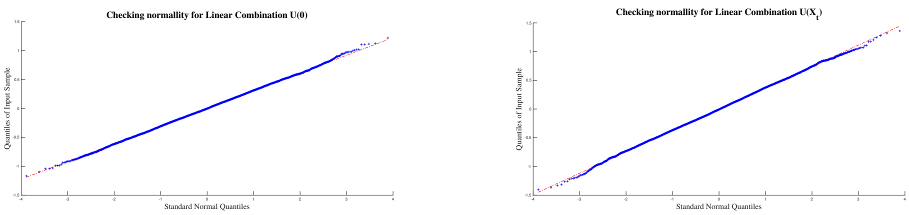

Figure 6: Q-Q plots for a linear combinations of components of $\mathbf{U}(\mathbf{0})$ and $\mathbf{U}(\mathbf{X}_t)$.

Remark 2.9. *Even though each component of* $\mathbf{U}(\mathbf{0})$ *and* $\mathbf{U}(\mathbf{X}_t)$ *is Gaussian and linear combinations of components of* $\mathbf{U}(\mathbf{0})$ *and* $\mathbf{U}(\mathbf{X}_t)$ *are Gaussian, according to Remark 2.8, we have no guaranties that* $\mathbf{U}(\mathbf{0})$ *and* $\mathbf{U}(\mathbf{X}_t)$ *are jointly Gaussian. In fact, in Figure 7, we have a Q-Q plot for a linear combination of random components of* $\mathbf{U}(\mathbf{0})$ *and* $\mathbf{U}(\mathbf{X}_t)$*. It is easily noticeable that data does not fit to the straight line corresponding to quantiles for a Gaussian distribution, which strongly suggest that* $\mathbf{U}(\mathbf{0})$ *and* $\mathbf{U}(\mathbf{X}_t)$ *are not jointly Gaussian.*

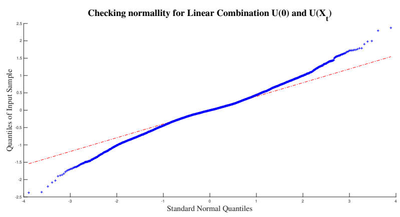

Figure 7: Q-Q plot for a linear combination of $\mathbf{U}(\mathbf{0})$ and $\mathbf{U}(\mathbf{X}_t)$.

3 Conclusion

In this work, we present a simple model for a scalar random field $S(\mathbf{x})$ written as a sum of Fourier modes. We obtain some of its statistical properties and prescribe conditions for which every sum of finitely many Fourier modes provides a scalar Gaussian field. As a consequence of the definition of incompressible fields, by taking the perpendicular gradient of $S(x)$ we obtain an incompressible bi-dimensional random field which components have the same distribution from the scalar field. This fact allows us to extend some theoretical results and obtain the statistical properties for the bi-dimensional random field. Additionally, we use contour plots to analyze how parameters in the model change the geometry of such scalar fields.

From this model for a stream function $S(\mathbf{x})$ we derive a model for a bi-dimensional random velocity field $\mathbf{U}(\mathbf{x})$, which is suitable to calculate the Taylor expansion for the Lagrangian velocity $\mathbf{U}(\mathbf{X}_t)$ and whose properties match with some important statistical properties of real turbulence, such as homogeneity, stationarity, and incompressibility. In addition, we prescribe conditions for which $\mathbf{U}(\mathbf{X}_t)$, for $t \geq 0$, is Gaussian. Furthermore, we design a numerical experiment and use Q-Q plots to illustrate some theoretical results concerning the distribution of components of $\mathbf{U}(\mathbf{0})$ and $\mathbf{U}(\mathbf{X}_t)$ and present graphical evidence that suggest $\mathbf{U}(\mathbf{0})$ and $\mathbf{U}(\mathbf{X}_t)$ are not jointly Gaussian.

Acknowledgements

The author was supported by a scholarship granted by the CAPES.

References

[1] M. Avellaneda and A. J. Majda. “Mathematical models with exact renormalization for turbulent transport”. In: **Communications in Mathematical Physics** 131.2 (1990), pp. 381–429.

[2] G. E. P. Box and M. E. Muller. “A note on the generation of random normal deviates”. In: **The Annals of Mathematical Statistics,** 29 (1958), pp. 610–611.

[3] R. A. Carmona and L. Xu. “Homogenization for time-dependent two-dimensional incompressible Gaussian flows”. In: **The Annals of Applied Probability** 7.1 (1997), pp. 265–279.

[4] S. Corrsin. “Atmospheric diffusion and air pollution”. In: **Advances in Geophysics,** 6 (1959), p. 161.

[5] F. W. Elliott and A. J. Majda. “Pair dispersion over an inertial range spanning many decades”. In: **Physics of Fluids,** 8.4 (1996), pp. 1052–1060.

[6] S. C. Port and C. J. Stone. “Random measures and their application to motion in an incompressible fluid”. In: **Journal of Applied Probability** 13.3 (1976), pp. 498–506.

[7] E. S. Schneider. “Exact calculations for the Lagrangian velocity”. PhD thesis. Bowling Green State University, 2019.

[8] E. S. Schneider and C. L. Zirbel. “Using symbolic expressions to get the Taylor expansion of the Lagrangian auto-covariance function”. In: **Proceeding Series of the Brazilian Society of Computational and Applied Mathematics** 8.1 (2021). DOI: 10.5540/03.2021.008.01.0504.

[9] G. I. Taylor. “Statistical theory of turbulence”. In: **Proceedings of the Royal Society of London.Series A, Mathematical and Physical Sciences** 151.873 (1935), pp. 421–444.

[10] W. A. Woyczynski. “Passive tracer transport in stochastic flows”. In: **Stochastic Climate Models,** 49 (2012), pp. 385–396.

[11] C. L. Zirbel. “Lagrangian observations of homogeneous random environments”. In: **Advances in Applied Probability** 33.4 (2001), pp. 810–835.

