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Single-Level Differentiability for Interval-valued Functions

Ulcilea A. S. Leal[1]
UFTM *Campus* Universitário de Iturama, Iturama, MG
Gino Maqui[2]
ICEN, Federal University of Pará, PA
Geraldo N. Silva
[3]UNESP Department of Applied Mathematics, São José do rio Preto, SP
Weldon Lodwick[4]
UCDenver Department of Applied Mathematics and Statistical Sciences, Denver, CO

Abstract. This study uses the theory of single-level difference for interval-valued functions to propose the concept of single-level differentiability, illustrate its calculations, and investigate how its single-level derivative (SL-derivative) relates to other mathematical derivatives.

Keywords. Interval space, C-difference, Intervalar-valued function, SL-derivative

1 Introduction

Interval Analysis is a form of numerical analysis that works with interval uncertainty. Researchers use this technique in abstract mathematical representations to model deterministic real-world phenomena.

In this groundbreaking study, Hukuhara [9] introduced the Hukuhara derivative of a set-valued mapping, and in 2009 Stefanini and Bede [17] built on this work to present a concept of differentiability based on the generalized Hukuhara difference (gH-difference for short). The concept of derivative is fundamental in determining real-valued functions, and when implementing Interval Analysis, one expects to be able to predict the derivative of an interval-valued function. However, although several researchers have applied this approach in their work (see [5, 6, 10, 17, 18]), Hukuhara's differentiability concept has a critical drawback: the paradoxical behavior of the solution of a set or a fuzzy differential equation.

In 2014, Chalco-Cano et al. [4] addressed this issue by introducing a new interval arithmetic called Single Level Constraint Interval Arithmetic (SLCIA). This arithmetic is a variant of the constraint interval arithmetic proposed by Lodwick [11] that operates with a single parameter in each interval operand of an expression. This equation results in a computationally simple interval arithmetic with many desirable properties not generally shared by interval operations previously defined in the literature. In our study, we understand differentiability through the lens of this arithmetic.

2 Basic concepts

In this section, we describe five basic elements of SLCIA as proposed in Chalco-Cano et al. [4]: constraint function, algebraic operations, equivalence with other operations, interval expression

[1]ulcilea.leal@uftm.edu.br
[2]ginomaqui@gmail.com
[3]gsilva@ibilce.unesp.br
[4]Weldon.Lodwick@ucdenver.edu

and the properties of the C-sum and C-difference.

Frist, Chalco-Cano et al. [4] stated that single-level constraint arithmetic is applied when the variation of values within the extremes of an interval occurs in tandem. This arithmetic assumes that variations between intervals are always in the same way.

Let $A = [\underline{a}, \overline{a}] \in \mathbb{I}(\mathbb{R})$. Then, a continuous function $A : [0, 1] \to \mathbb{R}$ such that

$$\min_{0\leq\lambda\leq1} A(\lambda) = \underline{a}, \quad \max_{0\leq\lambda\leq1} A(\lambda) = \overline{a},$$

will be called a constraint function associated with A. Also associated with the interval A is the decreasing convex constraint function A, defined by Chalco-Cano et al. [4] as $A : [0, 1] \to \mathbb{R}$ by means

$$A(\lambda) = \lambda\underline{a} + (1 - \lambda)\overline{a}, \quad 0 \leq \lambda \leq 1.$$

Remark 2.1. *In this paper, we define the decreasing convex constraint function. However, given that the decreasing and increasing convex constraint functions are analogous to each other, the same form can be used for the increasing convex constraint function. Different constraint functions can determine the same interval.*

Second, Chalco-Cano et al. [4] presented the algebraic operations on $\mathbb{I}(\mathbb{R})$ for the decreasing convex constraint function.

Definition 2.1. *[4] Let A and B be two intervals and let $*$ be an arithmetic operation on $\mathbb{R}$. Then,*

1. *the constraint function associated with interval $A \circledast B$ is given by $(A \circledast B)(\lambda) = A(\lambda) * B(\lambda)$, where $A(\lambda)$ and $B(\lambda)$ are the functions associated with A and B, respectively;*
2. *the single-level constraint arithmetic operation (C-operation for short) $A \circledast B$ on $\mathbb{I}(\mathbb{R})$ is given by the interval*

$$A \circledast B = \left[\min_{0\leq\lambda\leq1} (A(\lambda) * B(\lambda)), \max_{0\leq\lambda\leq1} (A(\lambda) * B(\lambda))\right],$$

provided that the minimum and maximum exist.

Example 2.1. *Let $A = [-1, 0]$ and $B = [1, 3]$ be two intervals. Then, the convex constraint functions associated with A and B are $A(\lambda) = -\lambda$ and $B(\lambda) = -2\lambda + 3$, respectively. So,*

$$A \ominus B = \left[\min_{0\leq\lambda\leq1} (\lambda - 3), \max_{0\leq\lambda\leq1} (\lambda - 3)\right] = [-3, -2].$$

Let $A, B \in \mathbb{I}(\mathbb{R})$ and $\alpha \in \mathbb{R}$ be given, then

1. $A \oplus B = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$.
2. $\alpha \odot A = [\min\{\alpha\underline{a}, \alpha\overline{a}\}, \max\{\alpha\underline{a}, \alpha\overline{a}\}]$.
3. $A \ominus B = [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}]$.

Additionally, Chalco-Cano et al. [4] showed that the C-sum of intervals and the C-multiplication of an interval by a scalar coincident with the usual operations of sum and multiplication by scalar. This element was denoted by Minkowski operations [7] and defined in Moore [13].

Futhermore, the C-difference, Markov's difference, and gH-difference are equivalent to each other (see [12, 16]). As such, Stefanini [16] introduced the gH-difference to guarantee that the difference between two intervals would always exist.

Chalco-Cano [4] further showed that $E(A_1, A_2, ..., A_n)$ is a correct expression in interval arithmetic if $E(x_1, x_2, ..., x_n)$ is correctly constructed in a formal language for arithmetic operations with real number operands $x_1, x_2, ..., x_n$ and usual arithmetic operations on real numbers (see [8]). In interval expressions, all real arithmetic operations $*$ have been replaced with $\circledast$.

Considering that $A_1(\lambda), ..., A_n(\lambda)$ are the convex constraint functions associated with $A_1, ..., A_n \in \mathbb{I}(\mathbb{R})$, the evaluation of a correct expression is performed according to the following rule:

$$E(A_1, A_2, ..., A_n) = \left[\min_{0\leq\lambda\leq 1} E(A_1(\lambda), ..., A_n(\lambda)), \max_{0\leq\lambda\leq 1} E(A_1(\lambda), ..., A_n(\lambda))\right].$$

This is the evaluated of the expression E with the given arguments provided that the min and max exist.

Proposition 2.1. *[4] If given two expressions $E_1(x_1, x_2, ..., x_n)$ and $E_2(x_1, x_2, ..., x_n)$, that have the same result, that is,*

$$E_1(x_1, x_2, ..., x_n) = E_2(x_1, x_2, ..., x_n) \text{ for all } x_1, x_2, ..., x_n \in \mathbb{R},$$

then

$$E_1(A_1, A_2, ..., A_n) = E_2(A_1, A_2, ..., A_n) \text{ for all } A_1, A_2, ..., A_n \in \mathbb{I}(\mathbb{R}).$$

An important fact to note is that $A \oplus (-A) = [0, 0]$ for any interval A.

Finally, Chalco-Cano [4] outline many properties of C-sum and C-difference such as associativity, commutativity, unique neutral element, and opposite element for the sum. For more information about C-operation and properties see Chalco-Cano [4].

The function $\parallel A \parallel := \max\{|\underline{a}|, |\overline{a}|\}$, where $A \in \mathbb{I}(\mathbb{R})$, satisfies

$$\begin{aligned}
&\parallel A \parallel = 0, \text{ if and only if } A = [0, 0];\\
&\parallel \alpha \odot A \parallel = |\alpha| \parallel A \parallel;\\
&\parallel A \oplus B \parallel \leq \parallel A \parallel + \parallel B \parallel;\\
&\parallel A \ominus B \parallel \leq \parallel A \parallel + \parallel B \parallel .
\end{aligned}$$

It is readily seen that the usual metric in $\mathbb{I}(\mathbb{R})$:

$$d(A, B) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}$$

is associated with the function $\parallel \cdot \parallel$ by $d(A, [0, 0]) = \parallel A \parallel$ and $d(A, B) = \parallel A \ominus B \parallel$.

This metric is equivalent with the metric d_C proposed by Chalco-Cano et al. [4] and the metric H proposed by Pompeiu-Hausdorff [15] between two intervals. The interval space with the metric Pompeiu-Hausdorff is a complete and separable metric space (see [1, 2]).

The interval space $\{\mathbb{I}(\mathbb{R}), \oplus, \odot, \ominus\}$ presents an interesting algebraic structure to develop an elementary calculus for interval-valued functions of a real variable.

3 Single-level derivative

In the context of the derivative of interval-valued functions, Stefanini and Bede [17] proposed the generalized Hukuhara differentiable based on the gH-difference. Bede and Gal [3] presented derivative concepts using the strongly and weakly generalized (Hukuhara) differentiable.

We defined a derivative by means of the C-difference as follows:

Definition 3.1. *Let $x_0 \in]a, b[$ and a real number $h \neq 0$ be given such that $x_0 + h \in]a, b[$, then the SL-derivative of a function $F : [a, b] \to \mathbb{I}(\mathbb{R})$ at x_0 is defined as*

$$D_{SL}F(x_0) = \lim_{h \to 0} \frac{1}{h} \odot (F(x_0 + h) \ominus F(x_0)). \tag{1}$$

If $D_{SL}F(x) \in \mathbb{I}(\mathbb{R})$ satisfying (1) exists, we say that F is a single-level differentiable (SL-differentiable for short) at x_0.

Given that the C-difference, gH-difference, π-difference, and M-difference are equivalent (see [12, 14]), it follows that the SL-derivative, gH-derivative, π-derivative, and M-derivative (see [12, 14]) are also identical.

Remark 3.1. *Considering $F(\lambda)(x_0 + h)$ and $F(\lambda)(x_0)$ are the convex constraint functions associated with $F(x_0 + h)$ and $F(x_0)$, respectively, we have*

$$F'(\lambda)(x_0) := \lim_{h \to 0} \frac{1}{h} (F(\lambda)(x_0 + h) - F(\lambda)(x_0)) \tag{2}$$

as a constraint function associated with the new interval. Then,

$$D_{SL}F(x_0) = \left[\min_{0 \le \lambda \le 1} F'(\lambda)(x_0), \max_{0 \le \lambda \le 1} F'(\lambda)(x_0) \right]$$

provided with the minimum and maximum exist.

Example 3.1. *Let $F : \mathbb{R} \to \mathbb{I}(\mathbb{R})$ be an interval-valued function such that $F(x) = A \odot x$, where A is an interval. If $A(\lambda)$ is the convex constraint function associated with the interval A, then $F(\lambda)(x) = A(\lambda)x$ and $F(\lambda)(x + h) = A(\lambda)(x + h)$ are the constraint functions associated with the intervals $F(x)$ and $F(x + h)$, respectively. Thus,*

$$\lim_{h \to 0} \frac{1}{h} (F(\lambda)(x + h) - F(\lambda)(x)) = \lim_{h \to 0} A(\lambda) = A(\lambda).$$

Therefore,

$$D_{SL}F(x) = \left[\min_{0 \le \lambda \le 1} A(\lambda), \max_{0 \le \lambda \le 1} A(\lambda) \right] = A.$$

Example 3.2. *If $F(x) = A \odot p(x)$, where p is a crisp differentiable function and A is an interval, then $D_{LS}F(x) = A \odot p'(x)$. If $A(\lambda)$ is the convex constraint function associated with the interval A, then $F(\lambda)(x) = A(\lambda)p(x)$ and $F(\lambda)(x + h) = A(\lambda)p(x + h)$ are the constraint functions associated with the intervals $F(x)$ and $F(x + h)$, respectively. Thus,*

$$\lim_{h \to 0} \frac{1}{h} (F(\lambda)(x + h) - F(\lambda)(x)) = A(\lambda)p'(x).$$

Therefore,

$$D_{SL}F(x) = \left[\min_{0 \le \lambda \le 1} A(\lambda)p'(x), \max_{0 \le \lambda \le 1} A(\lambda)p'(x) \right] = A \odot p'(x).$$

The next result expresses the SL-derivative in terms of the endpoint function derivative.

Theorem 3.1. *Let $F : T \to \mathbb{I}(\mathbb{R})$ be an interval-valued function such that $F(x) = [\underline{f}(x), \overline{f}(x)]$. If $\underline{f}$ and $\overline{f}$ are differentiable functions at $x_0 \in T$, then F is SL-differentiable at x_0 and*

$$D_{SL}F(x_0) = \left[\min_{0 \le \lambda \le 1} \{\lambda \underline{f}'(x_0) + (1 - \lambda)\overline{f}'(x_0)\}, \max_{0 \le \lambda \le 1} \{\lambda \underline{f}'(x_0) + (1 - \lambda)\overline{f}'(x_0)\} \right].$$

Example 3.3. *Let $F : \mathbb{R} \to \mathbb{I}(\mathbb{R})$ be an interval-valued function such that $F(x) = [e^{-x}, 2e^{-x}]$. In this context, $F'(x)(\lambda) = e^x(\lambda - 2)$ is the limit (2), and $e^x(\lambda-2) = \lambda(e^{-x})' + (1-\lambda)(2e^{-x})'$. Then,*

$$D_{SL}F(x) = \left[\min_{0\leq\lambda\leq 1}\{e^x(\lambda-2)\}, \max_{0\leq\lambda\leq 1}\{e^x(\lambda-2)\}\right] = [-2e^{-x}, -e^{-x}].$$

Note that, the converse of Theorem 3.1 is not true; that is, the SL-differentiability of F does not imply the differentiability of $\underline{f}$ and $\overline{f}$. Consider the example below.

Example 3.4. *Let $F : \mathbb{R} \to \mathbb{I}(\mathbb{R})$ be an interval-valued function such that $F(x) = [-|x|, |x|]$. This interval-valued function is SL-differentiable at $x = 0$ and $D_{SL}(0) = [-1, 1]$. However, $\underline{f}$ and $\overline{f}$ are not differentiable functions at $x = 0$, so $F'(\lambda)(x)$ is not differentiable for all $\lambda \in [0, 1]$.*
Indeed, let $F(\lambda)(x)$ be the convex constraint function associated with F, where

$$F(\lambda)(x) = \lambda(-x) + (1-\lambda)x.$$

So,

$$D_{SL}F(0) = \left[\min_{0\leq\lambda\leq 1}(-2\lambda+1), \max_{0\leq\lambda\leq 1}(-2\lambda+1)\right] = [-1, 1].$$

Therefore, $D_{SL}(0) = [-1, 1]$.

4 Conclusion

In this paper, we present the single-level differentiability concept for interval-valued functions and provid several examples of its application to better understand the theories laid out in this study. Future research on this topic should include the optimality condition for interval-valued optimization problems using the single-level derivative.

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