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# Asymptotic results for the Painlevé XXXIV equation

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When dealing with *linear* differential equations, some of the most basic yet powerful tools are integral transforms such as the Fourier or Laplace transforms. Amongst their core features, they transform the original differential operators into multiplication, which in practice means turning the differential evolution in the original variables into explicitly solvable and simpler models in the Fourier space.

However, the situation changes drastically when dealing with *nonlinear* differential equations, as in this case the powerful algebraic structures of the standard integral transforms become of little direct practical use due to the nonlinearities. Starting in the 20th century in the context of the NLS and KdV equations, and with developments still to our days, there has been a blooming of new ideas in the construction and exploration of nonlinear analogues of Fourier/Laplace transforms, within the framework that is known as *inverse scattering methods*.

A family of differential equations of utmost recent relevance, and to which inverse scattering methods apply, is the so-called family of *Painlevé equations*. When classifying all nonlinear second order ordinary differential equations whose only movable singularities are poles, Paul Painlevé came to a total of 60 equations, which were later reduced to a total of 6 equations and became known as the *Painlevé equations*.

With various developments in mathematical physics in the past 40 years, the Painlevé equations became a ubiquitous family that describes various different nonlinear phenomena in statistical mechanics. Remarkably, they play a central role in random matrix theory. Starting with a natural but general family of probability distributions on matrices, the theory of universality in random matrix theory tells us that statistics of such random matrices converge, in the large matrix size limit, to one of few universal laws, that depend on some symmetries of the original matrix distributions but not on specific features of the distributions themselves.

In [2], Its, Kuijlaars and Östensson studied a model of hermitian random matrices with unitary invariance, and found a new universality law described by a one-parameter family of limit kernels $K_\alpha$. These kernels, in turn, are expressed by particular solutions to the Painlevé XXXIV equation

$$u''(s) = 4u(s)^2 - 2su(s) + \frac{(u'(s))^2 - 4\alpha^2}{2u(s)}. \tag{1}$$

The solutions needed to construct $K_\alpha$ were first specified in [2] in terms of the so-called monodromy data, and were later in [1] also characterized by the same authors in terms of boundary values as $s \to \pm\infty$.

In this work, our major goal is to extend the asymptotic results of [1] into two directions. We consider a more general family of solutions $u(s)$ than considered in [1, 2], and obtain asymptotic formulas valid as $s \to \infty$ not only along the real axis but along more arbitrary directions in the complex plane.

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Our motivation comes from random matrix theory, as it was recently found [3] that a large family of multiplicative statistics of random matrix models are described by solutions to (1) other than the ones considered in [1, 2].

Our main tool is the isomonodromic/inverse scattering approach mentioned earlier. At its core, it involves the solution of a Riemann-Hilbert problem (shortly RHP), a boundary value problem in the theory of analytic complex function theory that we now describe.

Set $\Sigma = \mathbb{R} \cup [0, \mathrm{e}^{2\pi\mathrm{i}/3}\infty) \cup [0, \mathrm{e}^{-2\pi\mathrm{i}/3}\infty)$, orienting each arc of $\Sigma$ from the origin to $\infty$. For a certain explicit $2 \times 2$ matrix-valued function $J : \Sigma \to \mathbb{C}$, interpreted as initial data, the RHP asks for finding a $2 \times 2$ matrix-valued function $\Psi$ satisfying the following properties.

1. The entries of $\Psi$ are analytic on $\mathbb{C} \setminus \Sigma$.
2. The matrix $\Psi$ has continuous values $\Psi_\pm$ as $z$ approaches $\Sigma$ from its $\pm$-sides, and which are related by the jump condition
$$\Psi_+(z) = \Psi_-(z)J(z), \quad z \in \Sigma.$$
3. With $\sigma_3$ being the third Pauli matrix, $\Psi$ has asymptotic behavior
$$\Psi(z) = (I + O(1/z))z^{-\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\mathrm{e}^{-(\frac{2}{3}z^{3/2}+sz^{1/2})\sigma_3}, \quad z \to \infty. \tag{2}$$

We emphasize that the matrix $J$ above is an explicitly given data. It is piecewise constant as a function of $z$ but depends on certain parameters $(b_1, b_2, b_3, \alpha)$, called monodromy parameters, which are in general complex but satisfy certain constraints, the monodromy conditions. And as a remarkable fact, $J$ depends *linearly* on the $b_j$'s. So given monodromy parameters, the solution $\Psi = \Psi(\cdot \mid b_1, b_2, b_3, \alpha, s)$ also depends on these parameters, and additionally on the parameter $s$ appearing in (2).

The isomonodromy approach then says that solutions $\Psi(\cdot \mid b_1, b_2, b_3, \alpha)$ to the problem above are in one-to-one explicit correspondence with solutions $u(s \mid b_1, b_2, b_3, \alpha)$ to (1). With this at hand, our approach towards obtaining asymptotics of solutions $u(s \mid b_1, b_2, b_3, \alpha)$ consists of applying the nonlinear steepest descent method to the RHP above with, as said, monodromy parameters more general than the ones considered in the literature.

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## References

[1] A. R. Its, A. B. J. Kuijlaars, and J. Östensson. "Asymptotics for a special solution of the thirty fourth Painlevé equation". In: **Nonlinearity** 22.7 (2009), pp. 1523–1558.

[2] A. R. Its, A. B. J. Kuijlaars, and J. Östensson. "Critical edge behavior in unitary random matrix ensembles and the thirty-fourth Painlevé transcendent". In: **Int. Math. Res. Not. IMRN** 9 (2008), pp. 1–67.

[3] Guilherme L. F. Silva. "Contributions to the asymptotic theory of random particle systems and orthogonal polynomials". Instituto de Ciências Matemáticas e de Computação - Universidade de São Paulo. Habilitation thesis (livre-docência), Dec. 2021.


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