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On the Continuous-Time Complementarity Problem

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Abstract. This work deals with solving continuous-time nonlinear complementarity problems using the variational inequality problem. A relation is set up so that a stationary point of an unconstrained continuous-time programming problem is a solution for the continuous-time complementarity problem.

Key words. Complementarity, Variational Inequality, Continuous-time.

1 Introduction

Complementarity problems were firstly proposed as the question of finding an $n-$vector x which satisfies the system of inequalities

$$x \geq 0, \qquad Mx + b \geq 0 \qquad \text{and} \qquad x^\top(Mx+b) = 0 \tag{1}$$

where M is an $n \times n$ matrix, b is an $n-$vector of real numbers and "$\top$" denotes the transposition of vectors and matrices. Such problems are elegant generalizations of certain linear programming, quadratic programming, and game theory problems.

The importance of problem (1) lies in the fact that its form includes several problems by appropriate choices of the vector b and the matrix M. As examples of applications, we can cite the problem of the existence of solutions to linear programs (Cottle [2], Dorn [3]) that can be reduced to a problem in the format (1), the equilibrium point problem of bimatrix games (Lemke [8]) and the unloading problem for plane curves (Du Val [4]). For other examples of applications, see Isac [6].

The formulation (1) was expanded to include a broader class of problems such as nonlinear programming and was rewritten as the problem of finding an $n-$vector x which satisfies the system of inequalities

$$x \geq 0, \qquad f(x) \geq 0 \qquad \text{and} \qquad x^\top f(x) = 0 \tag{2}$$

where f is a mapping of $\mathbb{R}^n$ into itself. Among other authors who studied the formulation (2), Cottle [2] gave sufficient conditions for the existence of x, and Karamardian [7] established sufficient conditions for the existence of an unique solution.

Bodo and Hanson [1] extended the results of Karamardian [7] to the case where x is a bounded measurable function which maps some finite interval into $\mathbb{R}^n$, sufficient conditions for the existence of a unique solution were given and applications to continuous linear and nonlinear programming were presented.

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In this work, we propose to solve the continuous-time complementarity problem presented by Bodo and Hanson using the variational-type inequalities problem defined by Zalmai in [10]. In such a paper, Zalmai presents a generalized sufficiency criteria in continuous-time programming and uses it to study the existence of a solution for the variational-type inequalities problem.

The text is organized as follows. In Section 2, we define the continuous-time complementarity problem, the variational-type inequalities problem and we establish relationships between these problems. In Section 3, using the Fischer-Burmeister function [5], we derive an unconstrained equivalent problem in the sense that a stationary point of the unconstrained equivalent problem is a solution of the continuous-time complementarity problem. Final comments are given in Section 4.

2 Variational-Type Inequality Problem

The continuous-time complementarity problem is to find $x \in L_\infty([0,T];\mathbb{R}^n)$ such that, for a.e. $t \in [0,T]$, we have that

$$x(t) \in K, \qquad f(x(t),t) \in K^\circ \qquad \text{and} \qquad x(t)^\top f(x(t),t) = 0 \tag{3}$$

where $K \subset \mathbb{R}^n$ is a nonempty closed convex cone with vertex at 0, namely, if $x \in K$, $\alpha x \in K$ for all $\alpha > 0$. The polar cone K° of K is given by $K^\circ = \{y \in \mathbb{R}^n \mid y^\top x \geq 0,\ \forall\, x \in K\}$. $L_\infty([0,T];\mathbb{R}^n)$ denotes the Banach space of all Lebesgue-measurable essentially-bounded n-dimensional vector functions defined on the compact interval $[0,T] \subset \mathbb{R}$, with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1\leq i\leq n} \operatorname*{ess\,sup}_{t\in[0,T]} |x_i(t)|$$

and $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a nonlinear function with $f(\cdot,t)$ continuously differentiable throughout $[0,T]$ and $f(x,\cdot)$ measurable for each x.

Define the following subset of $L_\infty([0,T];\mathbb{R}^n)$:

$$\Omega = \{x \in L_\infty([0,T];\mathbb{R}^n) \mid x(t) \in K \text{ a.e. } t \in [0,T]\}.$$

Remark 2.1. *In this work we also consider degenerate solutions for (3), in other words, solutions such that $x_i(t) = 0$ and $f_i(x(t),t) = 0$ for some $t \in [0,T]$ and $i \in I = \{1,2,\dots,n\}$.*

Definition 2.1. *The Variational-type Inequality Problem $VIP(f,\Omega)$ consists in finding $x^* \in \Omega$ such that, for a. e. $t \in [0,T]$,*

$$\int_0^T f(x^*(t),t)^\top (x(t) - x^*(t))\, dt \geq 0,$$

for all $x \in \Omega$.

Lemma 2.1. *$x^* \in L_\infty([0,T],\mathbb{R}^n)$ is a solution of (3) if, and only if, $x^* \in L_\infty([0,T],\mathbb{R}^n)$ is a solution of the $VIP(f,\Omega)$.*

Proof. If $x^* \in L_\infty([0,T],\mathbb{R}^n)$ is a solution of (3) then $f(x^*,t) \in K^\circ$ and we can conclude, for a.e. $t \in [0,T]$, that

$$x(t)^\top f(x^*(t),t) \geq 0, \text{ for all } x \in \Omega. \tag{4}$$

Using (4) and the hypothesis, for a.e. $t \in [0,T]$ and for all $x \in \Omega$, we have that

$$\begin{aligned} f(x^*(t),t)^\top(x(t) - x^*(t)) &= f(x^*(t),t)^\top x(t) - f(x^*(t),t)^\top x^*(t) \\ &= f(x^*(t),t)^\top x(t) \\ &\geq 0, \end{aligned}$$

resulting that $x^* \in \Omega$ is a solution of $VIP(f,\Omega)$. Conversely, if $x^* \in \Omega$ is a solution of the $VIP(f,\Omega)$, then $x^*(t) \in K$ a.e. $t \in [0,T]$. The inequality in Definition 2.1 holds for all $x \in \Omega$. Particularly, for $x = 0 \in \Omega$ and $x = 2x^* \in \Omega$ we have that

$$\int_0^T f(x^*(t),t)^\top x^*(t)\, dt \le 0 \qquad \text{and} \qquad \int_0^T f(x^*(t),t)^\top x^*(t)\, dt \ge 0,$$

respectively, resulting in

$$\int_0^T f(x^*(t),t)^\top x^*(t)\, dt = 0. \tag{5}$$

Statement: For all $x \in \Omega$,

$$f(x^*(t),t)^\top x(t) \ge 0 \text{ a.e. } t \in [0,T].$$

Indeed, suppose that there exists $\tilde{x} \in \Omega$ and a subset $D \subset [0,T]$, with positive measure, such that $f(x^*(t),t)^\top \tilde{x}(t) < 0$ for all $t \in D$. Define $\bar{x} \in \Omega$ given by

$$\bar{x}(t) = \begin{cases} \tilde{x}(t) \ if \ t \in D, \\ 0 \quad if \ t \in [0,T] \setminus D. \end{cases}$$

Then, using (5) and the definition of $\bar{x}$, we have that

$$\begin{aligned} \int_0^T f(x^*(t),t)^\top (\bar{x}(t) - x^*(t))\, dt &= \int_0^T f(x^*(t),t)^\top \bar{x}(t)\, dt - \int_0^T f(x^*(t),t)^\top x^*(t)\, dt \\ &= \int_D f(x^*(t),t)^\top \tilde{x}(t)\, dt \\ &< 0, \end{aligned}$$

contradicting the fact that x^* is a solution of the $VIP(f,\Omega)$. Therefore, by the above statement, $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$, that is, $f(x^*(t),t) \ge 0$ a.e. $t \in [0,T]$, resulting from (5) that

$$f(x^*(t),t)^\top x^*(t) = 0 \text{ a.e. } t \in [0,T].$$

□

Now, let $x^* \in \Omega$ and define the auxiliary continuous-time problem

$$\begin{aligned} &\text{maximize} \quad P(x) = -\int_0^T f(x^*(t),t)^\top x(t)\, dt \\ &\text{subject to} \quad x \in \Omega. \end{aligned} \tag{6}$$

Proposition 2.1. *Let $x^* \in \Omega$ be arbitrary. If x^* is a solution of Problem (3) then $x^* \in \Omega$ is a global maximum point of Problem (6) with $P(x^*) = 0$. Conversely, if x^* is a global maximum point of Problem (6), then x^* is a solution of Problem (3).*

Proof. Suppose that x^* is a solution of (3). Then $x^* \in \Omega$, $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$ and

$$P(x^*) = -\int_0^T f(x^*(t),t)^\top x^*(t)\, dt = 0.$$

From $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$ it follows, for all $x \in \Omega$,

$$P(x) = -\int_0^T f(x^*(t),t)^\top x(t)\, dt \le 0.$$

Therefore, $P(x) \leq P(x^*)$ for all $x \in \Omega$ and x^* is a global maximum point of (6) with $P(x^*) = 0$. Conversely, if x^* is a global maximum point of (6), then $x^* \in \Omega$ and, for all $x \in \Omega$, we have that

$$P(x) \leq P(x^*) \Leftrightarrow \int_0^T f(x^*(t), t)^\top (x(t) - x^*(t))\ dt \geq 0,$$

implying that x^* is a solution of the $VIP(f, \Omega)$. From Lemma 2.1 we have that x^* is a solution of Problem (3). □

Remark 2.2. *If x^* is a global maximum point of Problem (6), noting that the gradient of inequality constraints are linearly independent, we have that Problem (6) satisfies all assumptions of Theorem 4.2 presented by do Monte and de Oliveira [9], that provides necessary optimality conditions for the nonlinear continuous-time optimization problem with equality and inequality constraints.*

The next example illustrates Proposition 2.1.

Consider Problem (3) with $x : [0, 1] \to \mathbb{R}$, $f(x(t), t) = [x(t)]^2 - x(t)$. If x^* is a global maximum point of (6), then Theorem 4.2 in [9] guarantees us that there exists $u^* \in L_\infty([0, 1]; \mathbb{R})$ such that, for a.e. $t \in [0, 1]$,

(i) $-f(x^*(t), t) + u^*(t) = 0 \Rightarrow u^*(t) = [x^*(t)]^2 - x^*(t)$,

(ii) $u^*(t) \geq 0$, $x^*(t) \geq 0$ and $u^*(t)x^*(t) = 0$,

resulting that $x^*(t) = 0$ or $x^*(t) = 1$ for a.e. $t \in [0, 1]$. As the constraints in Problem (6) are linear, we can conclude that the functions family $\{x_\rho\} \subset L_\infty([0, 1]; \mathbb{R})$, $0 \leq \rho \leq 1$, given by

$$x_\rho(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \rho, \\ 1, & \text{if } \rho < t \leq 1, \end{cases}$$

are global minimizers of (6) and, by Proposition 2.1, solutions of (3). Note that $x^* \equiv 0$ is a degenerate solution because $f(0, t) = 0$ while $x^* \equiv 1$ is a nondegenerate solution.

3 A Unconstrained Equivalent Problem

Let us consider K to be the positive octant of $\mathbb{R}^n$. In this case, $K = K^\circ$. The Fischer-Burmeister function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\varphi(a, b) = \sqrt{a^2 + b^2} - a - b.$$

This function has the property that $\varphi(a, b) = 0 \Leftrightarrow a \geq 0,\ b \geq 0,\ ab = 0$. As Problem (6) has no equality constraint, Theorem 4.2 in [9] guarantees that there exists $u^* \in L_\infty([0, T]; \mathbb{R}^n)$, $u^*(t) \geq 0$ a.e. $t \in [0, T]$, such that $\|f(x^*, t) - u^*(t)\| = 0$ and $\varphi(x_i^*(t), u_i^*(t)) = 0$, for a.e. $t \in [0, T]$ and $i = 1, \ldots, n$. Consider the following unconstrained continuous-time problem:

$$\text{maximize} \quad Q(x, u) = -\int_0^T F(x(t), u(t), t)\ dt \tag{7}$$

where x and u are functions in $L_\infty([0, T]; \mathbb{R}^n)$ and, for a.e. $t \in [0, T]$,

$$F(x(t), u(t), t) = \|f(x(t), t) - u(t)\|^2 + \sigma(x(t), u(t)),$$

where

$$\sigma(x(t), u(t)) = \sum_{i=1}^n [\varphi(x_i(t), u_i(t))]^2 \text{ a.e. } t \in [0, T].$$

Remark 3.1. *Note that the Fischer-Burmeister function is non-differentiable at* $(0,0)$. *But the function* σ *above is a sum of the squared Fischer-Burmeister functions and the differential of* φ^2 *at* $(0,0)$ *is zero.*

Theorem 3.1. $x^* \in L_\infty([0,T];\mathbb{R}^n)$ *is a solution of the problem (3) if, and only if, there exists* u^* *in* $L_\infty([0,T];\mathbb{R}^n)$ *such that* (x^*,u^*) *is a global optimal solution for the problem (7) with* $Q(x^*,u^*)=0$.

Proof. By definition of F, if $Q(x^*,u^*)=0$ then $F(x^*,u^*,t)=0$ a.e. $t\in[0,T]$. Therefore,

$$\begin{aligned} \sigma(x^*(t),u^*(t)) &= 0 \text{ a.e. } t\in[0,T] \\ &\Leftrightarrow \sum_{i=1}^{n}[\varphi(x_i^*(t),u_i^*(t))]^2 = 0 \text{ a.e. } t\in[0,T] \\ &\Leftrightarrow \varphi(x_i^*(t),u_i^*(t)) = 0 \text{ a.e. } t\in[0,T],\ i=1,\dots,n \\ &\Leftrightarrow x_i^*(t)\geq 0,\ u_i^*(t)\geq 0,\ x_i^*(t)u_i^*(t)=0 \text{ a.e. } t\in[0,T],\ i=1,\dots,n. \end{aligned}$$

Then, for a.e. $t\in[0,T]$, we have that $x^*(t)\in K$ and

$$\begin{aligned} f(x^*,t)-u^*(t)=0 &\Rightarrow f(x^*,t)=u^*(t)\geq 0 \Rightarrow f(x^*,t)\in K^\circ; \\ x^*(t)^\top f(x^*,t) &= x^*(t)^\top u^*(t) = \sum_{i=1}^{n} x_i^*(t)u_i^*(t) = 0. \end{aligned}$$

Therefore, x^* is a solution for Problem (3). Conversely, if x^* is a solution of Problem (3), by Proposition 2.1, x^* is a global maximum point of Problem (6). Using Remark 2.2, we conclude that $F(x^*,u^*,t)=0$ a.e. $t\in[0,T]$, in other words, (x^*,u^*) is a global maximum point of (7) with $Q(x^*,u^*)=0$. □

Most of the algorithms used in the resolution of Problem (7) guarantee convergence only to stationary points. A global minimum point is very hard to find. For this purpose, denote the $n\times n$ jacobian matrix of f with respect to its first argument evaluated at x by $Df(x,\cdot)$ and suppose that its entries belong to $L_\infty([0,T];\mathbb{R})$. We say that $p=(x,u)$ is a stationary point of (7) if, and only if, $\nabla F(p,\cdot)=0$.

Definition 3.1. *The* $n\times n$ *matrix* $M(x,t)$, $x\in\mathbb{R}$, $t\in[0,T]$, *with elements* $m_{ij}(x,t)$, $i,\ j=1,\dots,n$, *is positive definite at* $x^*\in L_\infty([0,T];\mathbb{R}^n)$ *if, for all* $y\in L_\infty([0,T];\mathbb{R}^n)$ *and a.e.* $t\in[0,T]$,

$$y(t)^\top M(x^*(t),t)y(t) > 0 \text{ whenever } y(t)\neq 0.$$

The next theorem relates stationary points of (7) to solutions of Problem (3).

Theorem 3.2. *Let* x^* *and* u^* *be functions in* $L_\infty([0,T];\mathbb{R}^n)$. *If* (x^*,u^*) *is a stationary point of (7) and* $Df(\cdot,t)$ *is definite positive at* x^*, *then* x^* *is a solution of (3).*

Proof. For a.e. $t\in[0,T]$, let $w^*(t)=f(x^*(t),t)-u^*(t)$,

$$\tilde{\varphi}(x^*(t),u^*(t)) = \begin{pmatrix} \varphi(x_1^*(t),u_1^*(t)) \\ \vdots \\ \varphi(x_n^*(t),u_n^*(t)) \end{pmatrix},$$

$$J(t) = \operatorname{diag}\left(\frac{\partial\varphi}{\partial x_i}(x_i^*(t),u_i^*(t))\right)_{i=1}^{n} \quad \text{and} \quad K(t) = \operatorname{diag}\left(\frac{\partial\varphi}{\partial u_i}(x_i^*(t),u_i^*(t))\right)_{i=1}^{n}.$$

If (x^*, u^*) is a stationary point of (7), then $\nabla F(x^*, u^*, t) = 0$ a.e. $t \in [0, T]$, that is,

$$Df(x^*(t), t)w^*(t) + J(t)\tilde{\varphi}(x^*(t), u^*(t)) = 0 \quad (8)$$
$$-w^*(t) + K(t)\tilde{\varphi}(x^*(t), u^*(t)) = 0 \quad (9)$$

From (8) and (9) we obtain

$$w^*(t)^\top Df(x^*(t), t)w^*(t) + w^*(t)^\top J(t)\tilde{\varphi}(x^*(t), u^*(t)) = 0 \quad (10)$$

and

$$w^*(t)^\top = \tilde{\varphi}(x^*(t), u^*(t))^\top K(t), \quad (11)$$

respectively. Using (10) and (11), we have that

$$w^*(t)^\top Df(x^*(t), t)w^*(t) = -\tilde{\varphi}(x^*, u^*)^\top \{K(t)J(t)\} \tilde{\varphi}(x^*(t), u^*(t)). \quad (12)$$

Noting that

$$\frac{\partial \varphi}{\partial x_i}(x_i^*(t), u_i^*(t))\frac{\partial \varphi}{\partial u_i}(x_i^*(t), u_i^*(t)) \geq 0 \text{ a.e. } t \in [0, T],\ i = 1, \ldots, n,$$

we have that the matrix $K(t)J(t)$ is positive semi-definite for a.e. $t \in [0, T]$, implying that $w^*(t)^\top Df(x^*, t)w^*(t) \leq 0$ a.e. $t \in [0, T]$. But, by hypothesis, $w(t)^\top Df(x^*(t), t)w(t) > 0$ for all $w \in L_\infty([0, T]; \mathbb{R}^n)$, whenever $w(t) \not\equiv 0$. Then

$$w^*(t) = 0 \text{ a.e. } t \in [0, T]. \quad (13)$$

By (13) and (9), for a.e. $t \in [0, T]$, we obtain

$$K(t)\tilde{\varphi}(x^*(t), u^*(t)) = 0 \Leftrightarrow \varphi(x_i^*(t), u_i^*(t))\frac{\partial \varphi}{\partial x_i}(x_i^*(t), u_i^*(t)) = 0,\ i = 1, \ldots, n.$$

So, for almost every t and for all $i = 1, \ldots, n$, we have that $\varphi(x_i^*(t), u_i^*(t)) = 0$ or $\frac{\partial \varphi}{\partial x_i}(x_i^*(t), u_i^*(t)) = 0$. If $\frac{\partial \varphi}{\partial x_i}(x_i^*(t), u_i^*(t)) = 0$, then $x_i^*(t) > 0$ and $u_i^*(t) = 0$, implying that $\varphi(x_i^*(t), u_i^*(t)) = 0$, $i = 1, \ldots, n$. Therefore, in both cases,

$$\tilde{\varphi}(x^*(t), u^*(t)) = 0 \text{ a.e. } t \in [0, T]. \quad (14)$$

Thereby, for a.e. $t \in [0, T]$, we have from (14) that $x_i^*(t) \geq 0$, $u_i^*(t) \geq 0$, $x_i^*(t)u_i^*(t) = 0$, $i = 1, \ldots, n$, resulting from (13) that $x^*(t) \in K$, $f(x^*(t), t) \in K^\circ$ and $x^*(t)^\top f(x^*(t), t) = 0$ for a.e. $t \in [0, T]$. $\square$

Considering Example 2, note that the solution x_ρ, $0 \leq \rho \leq 1$, along with the Lagrange multiplier $u_\rho(t) = 0$ a.e. $t \in [0, 1]$ are such that (x_ρ, u_ρ) is a stationary point for Problem (7). Indeed, from (8) and (9), we have that

- (i) $\{2x_\rho(t) - 1\}\{[x_\rho(t)]^2 - x_\rho(t) - u_\rho(t)\} + \left\{\frac{x_\rho(t)}{\sqrt{[x_\rho(t)]^2 + [u_\rho(t)]^2}} - 1\right\}\varphi(x_\rho(t), u_\rho(t)) = 0$;
- (ii) $-\{[x_\rho(t)]^2 - x_\rho(t) - u_\rho(t)\} + \left\{\frac{u_\rho(t)}{\sqrt{[x_\rho(t)]^2 + [u_\rho(t)]^2}} - 1\right\}\varphi(x_\rho(t), u_\rho(t)) = 0$;

For all $0 \leq \rho \leq 1$, with $u_\rho(t) = 0$ a.e. $t \in [0,1]$ and

$$x_\rho(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \rho, \\ 1, & \text{if } \rho < t \leq 1, \end{cases}$$

we have that (i) and (ii) hold. Then, by Theorem 3.2, x_ρ is a solution for Problem (3), for any $0 \leq \rho \leq 1$. We remember that if $x_\rho(t) = 0$ and $u_\rho(t) = 0$, then $\nabla(\varphi(x_\rho(t), u_\rho(t)))^2 = 0$.

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