Proceeding Series of the Brazilian Society of Computational and Applied Mathematics
Preprint

Nonlinear normal modes of nonuniform flexible beams

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Flexible beams contain geometric nonlinearities emanated from the large displacements and large rotations of the cross sections. When the geometry of the beam is nonuniform, the equation of motion becomes complicated to derive, but can be efficiently approximated using the co-rotational finite element method. This paper proposes a procedure to compute nonlinear normal modes (NNM) of nonuniform flexible beams. The Rosenberg's definition of NNM is applied. The periodic solutions are computed using the Harmonic Balance Method (HBM) and the continuation of the modes properties with respect to the energy level is performed using the arc-length method. Examples of clamped-clamped beams with different cross sections variations are presented, illustrating the respective impacts in the NNMs.

Keywords. Nonuniform flexible beams, Nonlinear normal modes, Co-rotational finite element, Harmonic Balance Method

1 Introduction

For several industries, the dimensions of important structures are being extended to new limits in order to satisfy their new challenging needs. Those new dimensions led many structures to exhibit significant nonlinearities in their motion. A typical example of nonlinear structures is the offshore pipelines that are being used to explore oil at deep see levels [1]. The extended length of the pipes turn them into extreme slim beams with significant flexibility. The large displacements and finite rotations of the beam's cross sections contribute then for the generation of a geometric non-linearity in the structure model.

Flexible beams are usually analyzed under the assumption of small strains. This hypothesis allows this type of structure to be accurately modeled with co-rotational finite elements. The main idea of the co-rotational formulation is to decompose the motion of each element as a small elastic deformation added to a rigid body motion. A local coordinate system is then incorporated to each element and forced to move with it according to its rigid body part of the motion. Then, the small deformation is written with respect to this local coordinate system (using typical linear beam elements), and latter transformed to an inertial frame considering the rigid body motion. This transformation between frames generates the geometric nonlinear terms in the model that are associated with the large displacements and rotations of the beam's cross sections. The equation of motion is then formulated in the inertial coordinate system using the Lagrangian equation.

The novelty in this paper corresponds to the computation of nonlinear normal modes (NNM) of nonuniform flexible beams. Previous publication of flexible beams modeled with co-rotation beam elements were restricted to numerical integration of the equation of motion. However, as highlighted in [5], the knowledge of NNMs allows a thoroughly understanding of a system's vibratory response in the nonlinear regime. Here, the Rosemberg's definition of NNM is used. He defined

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the NNMs as synchronous oscillations of the system, which was later generalized to periodic solutions (non-necessarily synchronous oscillation) to account mode interactions (internal resonances). In the literature, there are many advanced numerical methods that compute periodic solutions of mechanical systems. The Harmonic Balance Method (HBM) is perhaps the most popular one [6]. It approximates the solution of a periodic boundary value problem using a truncated Fourier series as an *Ansatz* that satisfies a weak formulation. For a complete computation of the NNMs, the HBM is then combined with a continuation method that evaluates the periodic solutions for different energy levels [2].

2 Equation of motion

Several researches have already analyzed the dynamic responses of flexible beams using co-rotational finite elements [3, 4]. The main differences between them lie in the choice of the interpolation functions used in the local frame when computing the kinetic and potential energies of the system. In this paper, linear interpolation is used to derive the inertial terms (Timoshenko elements), while cubic interpolation is used to derive the elastic terms (IIE element). A complete description on how to obtain the equation of motion is given in [8], and it results in

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{f}\left(\mathbf{q}\right) = \mathbf{0}, \tag{1}$$

where $\mathbf{M} \in \mathbb{R}^{n\times n}$ is the mass matrix, $\mathbf{f} \in \mathbb{R}^n$ is the vector with nonlinear elastic forces, $\mathbf{q} \in \mathbb{R}^n$ is the vector of generalized coordinates and n is the number of degrees of freedom.

Instead of performing a numerical integration of this equation of motion, as already performed in previous publications, this paper incorporates this co-rotational model in a procedure to solve periodic boundary value problems.

3 Nonlinear normal modes

For the computation of the NNMs of nonuniform flexible beams, the Rosemberg's definition will be used here. It consists in solutions of periodic boundary value problems of an autonomous systems. Given the dependency of energy in the responses, NNMs become also depend on the energy level. Therefore, the periodic solutions must be computed for a pre-defined range of energies, which can be done combining the Harmonic Balance Method (HBM) [6] with the arc-length continuation method (which is based on a prediction-correction scheme)[7].

The HBM is a popular method used to solve periodic boundary value problems. For the flexible beam considered in this paper, the periodic boundary value problem can be constructed adding a periodic boundary restriction to the equation of motion, which leads to:

$$\begin{aligned} &\text{Solve:} && \mathbf{M}\ddot{\mathbf{q}} + \mathbf{f}\left(\mathbf{q}\right) = \mathbf{0}, \quad t \in [0, T] \\ &\text{With:} && \mathbf{q}(0) = \mathbf{q}(T) \\ & && \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}(T). \end{aligned} \tag{2}$$

Here, $T = \frac{2\pi}{\Omega}$ is the unknown fundamental period of the solution. Instead of seeking the periodic solution of Eq. (2) directly, the HBM starts with the definition of an *Ansatz* function $\mathbf{q}_H(t)$, written as a truncated Fourier series, that converges to $\mathbf{q}(t)$ as the truncation order increases. The *Ansatz* is defined as

$$\mathbf{q}_H(t) = \sum_{k=-H}^{H} \mathbf{Q}_k e^{\mathrm{i}k\Omega t}, \tag{3}$$

where H is the truncation order of the series, $\{\mathbf{Q}_k\}_{k=-H}^{H} \in \mathbb{C}^n$ are the respective Fourier coefficients of the *Ansatz* and Ω is the fundamental frequency. The vector basis used to span the *Ansatz* corresponds to a set of Fourier functions, which are intrinsically periodic, and therefore automatically satisfy the periodic boundary conditions required by Eq. (2). Since the *Ansatz* is only an approximation of $\mathbf{q}(t)$, a residual $\mathbf{r}(t)$ is expected when introducing Eq. (3) into the equation of motion:

$$\mathbf{r}(t) := \sum_{k=-H}^{H} -\omega^2 k^2 \mathbf{M}\mathbf{Q}_k e^{\mathbf{i}k\Omega t} + \mathbf{f}\left(\mathbf{q}_H\right) \neq \mathbf{0}. \tag{4}$$

Since the nonlinear elastic force vector, $\mathbf{f}\left(\mathbf{q}_H\right)$, is a function of the *Ansatz* only, it is also periodic. Therefore, the residual can be expressed as

$$\mathbf{r}(t) = \sum_{k=-\infty}^{\infty} \mathbf{R}_k \left(\Omega, \{\mathbf{Q}_l\}_{l=0}^{H}\right) e^{\mathbf{i}k\Omega t}, \tag{5}$$

where $\mathbf{R}_k$ corresponds to the k-th Fourier coefficient of the residual.

When projecting the residual into the subspace of the *Ansatz* (performing a Fourier-Galerking projection), the time dependency of Eq. (5) is removed. Also, using the orthogonality of the Fourier functions and imposing that the residual must be perpendicular to the subspace of the *Ansatz* (i.e., balanced up to the H-th harmonic), a system of nonlinear algebraic equation is constructed

$$\mathbf{R}_m \left(\Omega, \{\mathbf{Q}_l\}_{l=0}^{H}\right) = \mathbf{0} \quad \text{for} \quad m = -H, \dots, H. \tag{6}$$

Since the Fourier coefficients $\mathbf{R}_m$ shares the conjugate mirror property, it is sufficient to solve Eq. (6) only for $m = 0, \dots, H$.

This system of nonlinear algebraic equations is under-determined and requires two additional equations, one related to a phase restriction and other to an amplitude restriction. A popular phase restriction consists in

$$\eta_p \left(\{\mathbf{Q}_l\}_{l=0}^{H}\right) := \sum_{l=1}^{H} l \mathbf{c}_i^T \Im\left\{\mathbf{Q}_l\right\} = 0, \tag{7}$$

where $\mathbf{c}_i$ is an unit vector with all components equal to zero, except the i-th entry. This equation imposes zero initial velocity in the i-th degree of freedom, and therefore restricts the phase of the periodic solution. Regarding the amplitude normalization, a modal mass normalization considering all harmonics is adopted here [6]:

$$\eta_a \left(\{\mathbf{Q}_l\}_{l=0}^{H}, \epsilon\right) := \sum_{l=0}^{H} \left(\mathbf{Q}_l^*\right)^T \mathbf{M}\mathbf{Q}_l - \epsilon, \tag{8}$$

where ϵ is the modal mass, a parameter that is related to the desired energy in the system. This important parameter will be used later as free parameter in the continuation method to compute the branches of periodic solutions.

The combination of Eq. (6) with Eq. (7) and Eq. (8) leads to the nonlinear system of algebraic equation with a unique solution. It can be written in a compact form as

$$\mathcal{R}(\mathbf{x}, \epsilon) = \begin{bmatrix} \mathbf{R}_0(\mathbf{x}) \\ \Re\{\mathbf{R}_1(\mathbf{x})\} \\ \Im\{\mathbf{R}_1(\mathbf{x})\} \\ \vdots \\ \Re\{\mathbf{R}_H(\mathbf{x})\} \\ \Im\{\mathbf{R}_H(\mathbf{x})\} \\ \eta_p(\mathbf{x}) \\ \eta_a(\mathbf{x}, \epsilon) \end{bmatrix} = \mathbf{0}. \tag{9}$$

where $\mathbf{x} = [\Omega, \mathbf{Q}_0, \Re\{\mathbf{Q}_1\}, \Im\{\mathbf{Q}_1\}, \ldots, \Re\{\mathbf{Q}_H\}, \Im\{\mathbf{Q}_H\}]$ are the vector of unknowns. The solution of Eq. (9) is usually found numerically using a Newton type methods.

The system of nonlinear algebraic equations defined in Eq. (9) gives a good approximation of the periodic solution for a specific energy level. To completely characterize the NNMs, the periodic solution must be evaluate for a wide range of energy, i.e., considering $\epsilon \in [\epsilon_{min}, \quad \epsilon_{max}]$. This can be efficiently done using a numerical path continuation and using ϵ as free parameter. The set of solutions at different energy levels creates a periodic solution branch. The modal mass ϵ becomes an additional unknown parameter of the problem, and must be found in the solution of Eq. (9) with a parametric restriction. In this paper, the predictor-corrector scheme was chosen as continuation method. The tangent method was used for the prediction step and the arc-length for the correction step. A complete description of this continuation method is given in [7].

4 Numerical examples

Clamped-clamped flexible beams with nonuniform cross sections were chosen to exemplify the proposed procedure for computing NNNMs of nonuniform flexible beams. In total, two configurations of flexible beams are considered: one having the thickness and height of the cross section decreasing linearly from the edges to the center (Beam 1), and the other increasing from the edges to the center (Beam 2). The geometric and material properties of both beams are presented in Fig. 1. An illustration of the adopted mesh for the Beam 1 is presented in Fig. 2. It covers only half of the beam's domain since the symmetry in the structure is consider. The equations of motion of the beams were obtained using the co-rotational formulation. The first NNM was computed using the HBM with truncation order $H = 6$. The Alternating Frequency-Time (AFT) was used to compute the nonlinear part of the Fourier coefficients of the residue. The Newton-Raphson method was used to solve the nonlinear algebraic equations that characterize the periodic approximation. The numerical continuation was carried out for a mechanical energy varying from 10^{-3} J to 10^{3} J.

The fundamental frequency of the periodic approximations are presented in the frequency-energy plot (FEP) in Fig. 3. For both beams, a stiffening behavior is observed since the fundamental frequencies increase with the energy level. The fundamental frequency of Beam 2 is significantly lower than the fundamental frequency of Beam 1, mainly because more mass is concentrated at lower stiffness points of the beam, i.e., close to the middle. This is more expressive for low energy levels, where the ratio between the fundamental frequencies is approximately 2. However, at the maximum energy level, the same ratio becomes approximately 1.25, which means that values of the fundamental frequencies became relatively closer to each other.

The motion of the NNMs at two particular energy levels (10^{-2} J and 10^{2} J) is further analyzed and presented in Fig. 4 for Beam 1, and in Fig. 5 for Beam 2. The Fourier coefficients of the motions at the middle point and in the quarter point of the beams (represented by the blue and red lines, respectively) are also presented. At the low energy levels, the motion of the clamped-clamped

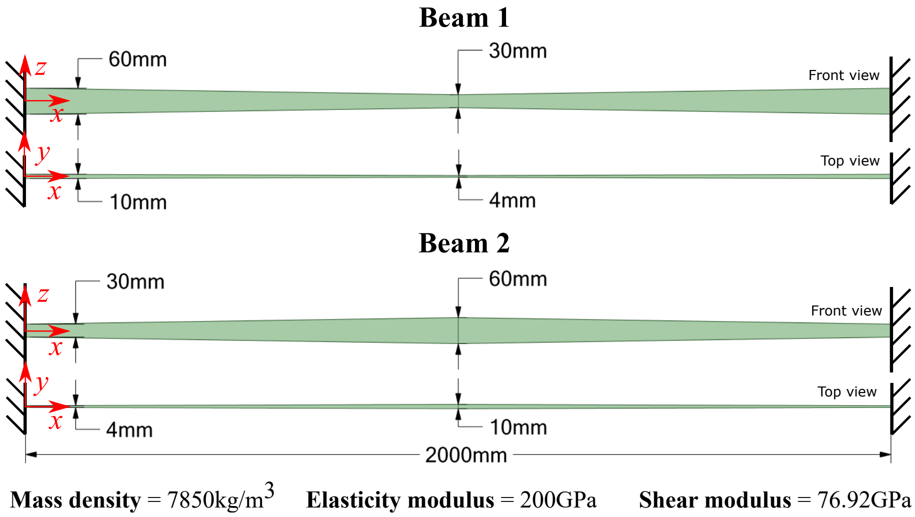

Figure 1: Material and geometric properties of the nonuniform flexible beams.

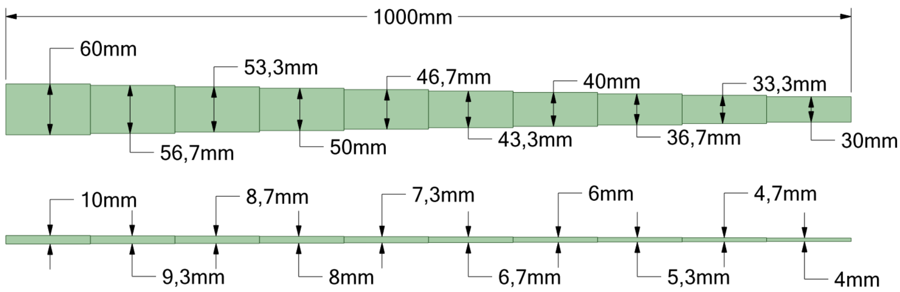

Figure 2: Adopted mesh for Beam 1 considering the symmetry of the structure.

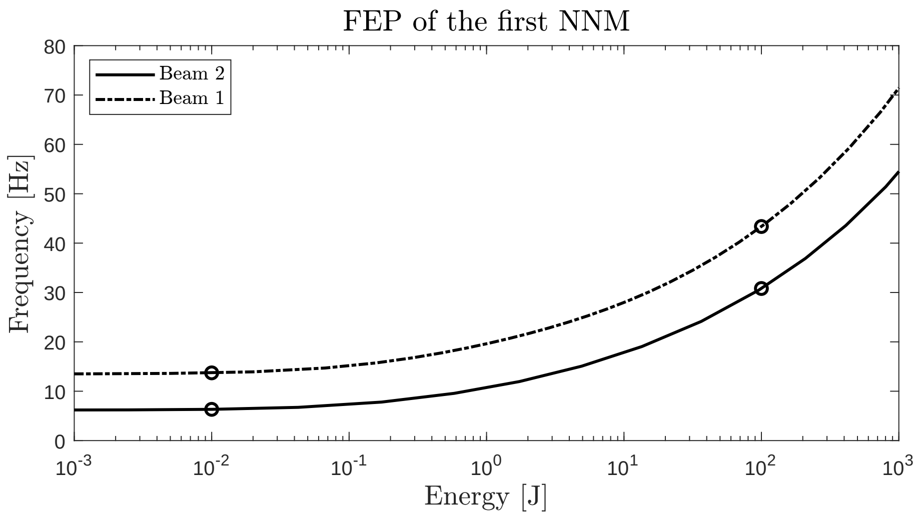

Figure 3: Frequency-energy plot of the first NNMs of Beam 1 and Beam 2.

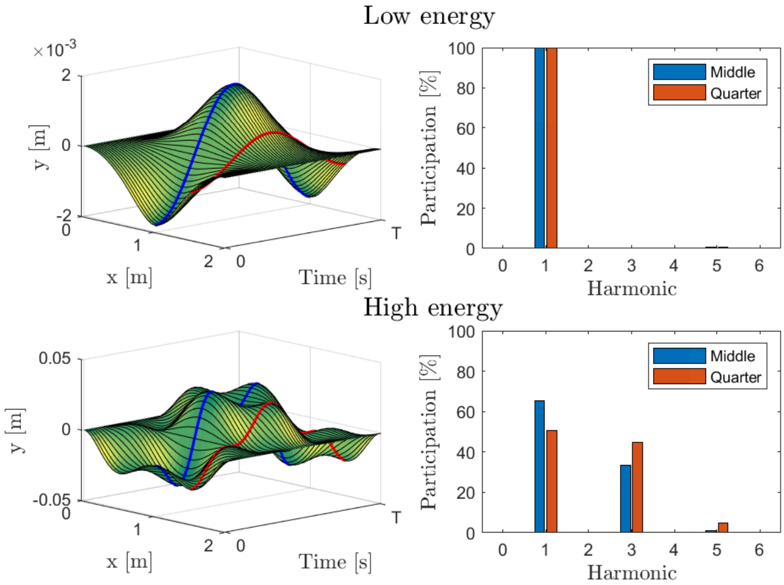

Figure 4: NNM motion of Beam 1 at low and high energies. Fourier coefficients of the motion at middle and one quarter of the beam.

nonuniform flexible beams is approximately equal to the linear normal modes of the underlying linear system, where only the fundamental harmonic defines the motion. However, at high energy levels, the NNM motion of Beam 1 exhibits significant participation of higher harmonics when compared to the NNM motion of Beam 2 at the same energy level. Therefore, it is possible to conclude that the participation of the higher harmonics in the NNMs motions can be reduced by changing the distribution of material along the beam.

5 Conclusions

Flexible beams were modeled here using the co-rotation finite element method. Models with geometric nonlinearities emanated from the large displacements and rotations of nonuniform cross sections could be efficiently computed with this method. The equations of motion were used to construct periodic boundary value problems that determine the NNMs. The solutions were successfully computed combining the HBM with the arc-length continuation. From the results of the presented example, it was possible to conclude that clamped-clamped flexible beams present a stiffening behavior since the fundamental frequency increases with the energy level. Also, the participation of the higher harmonics in the NNM motion could be reduced adopting a different distribution of material along the beam, which could be consider to improve the design of flexible beams vibrating at high energy levels.

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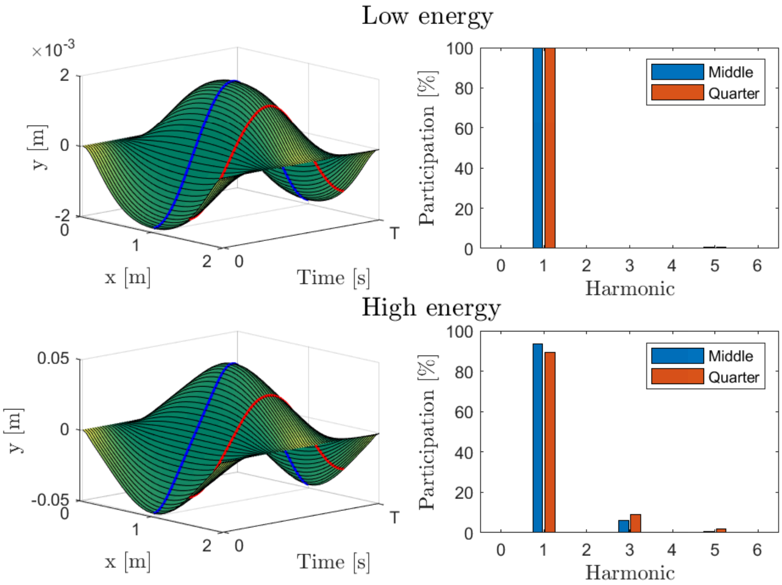

Figure 5: NNM motion of Beam 2 at low and high energies. Fourier coefficients of the motion at middle and one quarter of the beam.

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