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Resonant Orbits due to Evection Resonance

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Abstract.

Depending on the frequency of the mean motion of the third body, some secular resonances involving this frequency and the frequencies of the longitude of the pericenter or longitude of the ascending node of the satellite, can appear in this problem. In this case, averaging in the mean motion of the Sun can not be applied, while single averaging in the mean longitude of the satellite is always recommended since it greatly simplifies the calculations. We developed the equations to calculate the resonant semimajor axis due to the evection resonance considering the J_2 and C_{22} terms. With these equations, we show some examples of the resonant semimajor axis for the planets Earth and Mercury and of the dwarf planet Haumea. Using the single-averaged model (keeping only J_2 and third body perturbation), we draw some simple curves (Contour Plots) to identify resonant orbits due to evection resonance.

keywords. Astrodynamics, Non-Linear Dynamic, Evection Resonance, Artificial Satellite.

1 Introduction

In addition to classical mean motion resonance (MMR), there exists another very important group of resonances which involves angles whose periods are much longer when compared to single fast orbital periods like those seen in MMR. These resonances usually involve the combination of the longitude of the pericenters or longitude of the nodes and sometimes the mean longitude of a distant (but very massive) third body. We call secular resonances as they are connected to very long periods when compared do MMR. Kozai and evection resonance are some typical cases and they play important role not only in our solar system but also in the dynamics of the extrasolar celestial bodies. Here, in this work, we analyze the effect of the evection resonance. Analytical expressions are developed in this case. Analysis of the orbital motion of an artificial satellite around planets or moons are presented taking into account the non sphericity of the primaries and the perturbations coming from a third body in an elliptical and inclined orbit. Depending on the frequency of the mean motion of the third body, some secular resonances involving this frequency and the frequencies of the longitude of the pericenter or longitude of the ascending node of the satellite, can appear in this problem. In this case, averaging in the mean motion of the Sun can not be applied, while single averaging in the mean longitude of the satellite is always recommended since it greatly simplifies the calculations. However, some caution is necessary in some cases. Even in our solar system we have some bodies where the C_{22} coefficient is almost of the same order of

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the classical J_2 (Mercury, Moon, Haumea, etc). We developed a new equation to calculate the resonant semimajor axis when the C_{22} term is taken into account.

The classical problem of the critical inclination certainly is one of the oldest problems of secular resonance in the study of dynamics of artificial satellite. In [7] the author discusses the importance of the small divisors that arise in the vicinity of the critical inclination. The secular resonance effect was used to show that the growth in the eccentricity, as observed in spatial debris located in the MEO region at inclination approximately equal to 56 degrees, can be explained as a natural effect of the secular resonance $2\dot{w}+\dot{\Omega}=0$. See also similar investigation about this kind of resonance in [9]. In this work, the authors emphasize the importance of the inclination of the Moon, and the also the fact that this resonance does not depend on the semimajor axis of the satellite. Some optimal initial conditions for the pair (ω,Ω) that provide maximum or minimum increase of the eccentricity are studied. More recently, in [4] is shown an interesting up date discussion of this important resonance which involves the debris problems. The authors show that the growth of eccentricity caused by resonance $2\dot{w}+\dot{\Omega}=0$ can be used as an effective strategy for moving space debris into non-operative or graveyard orbits. An approach to investigate the probability of capture in evection resonance as a function of the tidal evolution rate and initial eccentricity is presented in [12]. Analytical expressions are developed for analysis of the resonant system, and the escape mechanism of the evection resonance is explored. In [6], the effect of the evection resonance is strongly emphasized in the study of the stability of natural satellites of Saturn. In [11], the effects of the evection resonance are shown to be crucial in the survival of the satellites of some exoplanets when migrating inward to the star. By evection resonance we mean a 1:1 commensurability between the mean motion of exoplanet (primary) around a star (third body) and the longitude of the pericenter of the satellite around the exoplanet. We developed the equations to calculate the resonant semimajor axis due to the evection resonance considering the J_2 term (for the classic term due to J_2 see [14, 15]) and a equation considering the C_{22} term. With these equations, applications are made to calculate the resonant semimajor axis of the planets Earth and Mercury and of the dwarf planet Haumea. Considering the resonant disturbing potential, obtained from the single-averaged model, due to evection resonance, diagrams are presented using Contour Plots to identify resonant orbits.

2 Evection resonance

Considering a low altitude artificial satellite, and assuming that J_2 is the dominant term, we first restrict the non-sphericity of the primary to only the portion due to the $< R_{J2} >$ term as given in equation (12) of the reference [3]. From Lagrange's planetary equations we have:

$$\frac{dg}{dt}=\frac{3}{4}\,\frac{nJ_2\,R^2\big(5\,\cos^2(i)-1\big)}{(e^2-1)^2a^2} \tag{1}$$

$$\frac{dh}{dt}=-\frac{3}{2}\,\frac{nJ_2\,R^2\cos(i)}{(e^2-1)^2a^2} \tag{2}$$

The orbital parameters of the spacecraft are (a,e,i,g,h,n) semimajor axis, eccentricity, inclination, argument of the pericenter, longitude of the node and mean motion, respectively. In the evection resonance, the 1:1 commensurability between ϖ and $\lambda_\odot$, where $\lambda_\odot$ is the mean longitude of the third body (Sun) and $\varpi=g+h=$ longitude of the pericenter of the satellite, is characterized when we have:

$$2\dot{g}+2\dot{h}-2n_\odot=0 \tag{3}$$

where $n_\odot$ is the mean motion of the third body as seen from the primary that hosts the satellite. Here, in our case $n_\odot$ is constant, since the motion of the third body is assumed to be known. Replacing equations (1) and (2) in equation (3) and solving for a, the solution is given by

$$a_{res}{}^{7/2} = \frac{3}{4}\frac{\sqrt{\mu}J_2 R_p{}^2\left(5\,(\cos(i))^2-2\,\cos(i)-1\right)}{(1-e^2)^2 n_\odot} \tag{4}$$

It is known that some celestial bodies have the C_{22} term (due to equatorial ellipticity) almost of the same order of magnitude as J_2. Thus, if we consider, in addition to $< R_{J2} >$, the $< R_{C22} >$ term given by equation (14) of the reference [3] and to replace in Lagrange's planetary equations, we get

$$\frac{dg}{dt} = -\frac{3}{4}\frac{\mu\left(10\,\cos(2\,h)(\cos(i))^2 C_{22}\,R^2-5\,(\cos(i))^2 J_2\,R^2-6\,C_{22}\,R^2\cos(2\,h)+J_2\,R^2\right)}{na^5(e^2-1)^2} \tag{5}$$

$$\frac{dh}{dt} = \frac{3}{2}\frac{\cos(i)\left(2\,C_{22}\,R^2\cos(2\,h)-J_2\,R^2\right)\mu}{na^5(e^2-1)^2} \tag{6}$$

Replacing equations (5) and (6) in equation (3) and solving for a, the solution is given by

$$a_{res}{}^{7/2} = \frac{3}{2}\,\frac{\sqrt{\mu}R^2\left(\left(-5\,(\cos(i))^2+2\,\cos(i)+3\right)C_{22}\,\cos(2\,h)+1/2\left(5\,(\cos(i))^2-2\,\cos(i)-1\right)J_2\right)}{(1-e^2)^2 n_\odot} \tag{7}$$

Now, just to see the main feature of the phase space of the evection resonance, we consider some crude approximations: we take only the secular terms and the resonant cosine, that is, the term factored by $\cos(2g + 2h - 2\lambda_\odot)$, obtained from the equation of the disturbing potential due to the third body (Sun) in an elliptical and inclined orbit:

$$\begin{aligned} R2_{res} = \tfrac{27\mu' n_\odot{}^2 a^2}{16}((A^2 + B^2 + C^2 + D^2 + 2E^2 - 2/3)(e_\odot{}^2 + 2/3)(e^2 + 2/3)- \\ \tfrac{25}{9}C^2(e_\odot{}^2 - 2/5)e^2\cos(2g + 2h - 2\lambda_\odot)) \end{aligned} \tag{8}$$

Since $\lambda_\odot = n_\odot t + \theta_\odot$, then this potential depends explicitly on time. In order to continue, a convenient change to Delaunay canonical elements is very convenient, since we can reduce this problem to a one degree of freedom conservative Hamiltonian problem. This can be easily done in the following way ([15], [1], [8]):

$$\begin{aligned} R2_{res} = \tfrac{27\mu' n_\odot{}^2 a^2}{16}((A^2 + B^2 + C^2 + D^2 + 2E^2 - 2/3)(e_\odot{}^2 + 2/3)(e^2 + 2/3)- \\ \tfrac{25}{9}C^2(e_\odot{}^2 - 2/5)e^2\cos(2\varpi - 2\lambda_\odot)) - n_\odot P_\odot \end{aligned} \tag{9}$$

For the moment, we have $R2_{sec} = R2_{sec}(a, e, i, g, h, \lambda_\odot)$. Writing $R2_{sec}$ in terms of the classical Delaunay variables (L, G, H, l, g, h), (Brouwer & Clemence,[1]) we have: $R2_{sec(L,G,H,--,g,h,\lambda_\odot)}$. As l = mean anomaly has been eliminated, L = is constant, so that the effective hamiltonian of this problem is: $H2_{sec} = R2_{sec(G,H,g,h,\lambda_\odot)}$. In order to work with conservative Hamiltonian, we extend the phase space, so that the Hamiltonian in the extended space now writes:

$H2_{sec*} = H2_{sec}\breve{}\,n_\odot * P_\odot$, where $P_\odot$ is the conjugated momentum of $\lambda_\odot$, so that we have $(G, H, P_\odot, g, h, \lambda_\odot)$ as the new set canonical variables. Now, instead of classical Delaunay, we prefer to move to slow Delaunay variables, which are also canonical variables (Brouwer & Clemence,[1]), that is: $G - L \to \varpi$; $H - G \to h$; $P_\odot \to$; $\lambda_\odot$. Now we perform the last trivial transformation:

$(G - L, H - G, P_\odot, \varpi, h, \lambda_\odot) \to (P_1, P_2, P_3, \theta_1, \theta_2, \theta_3)$, where we take

$\theta_1 = \lambda_\odot$; $\theta_2 = h$; $\theta_3 = \varpi - \lambda_\odot$. Now, taking into account the condition of Jacobi-Poincaré $(G-L)d\varpi + (H-G)dh + P_\odot d\lambda_\odot = P_1 d\theta_1 + P_2 d\theta_2 + P_3 d\theta_3 = P_1 d\lambda_\odot + P_2 dh + P_3 d(\varpi - \lambda_\odot)$. Thus, we get, $G - L = P_3$, $H - G = P_2$, $P_\odot = P_1 - P_3$, $H = P_2 + G$, $H = P_2 + P_3 + L$. Therefore, the $R2_{res}$ can be written in the form

$$\begin{aligned} R2_{res} = \tfrac{27\mu' n_\odot{}^2 a^2}{16}((A^2 + B^2 + C^2 + D^2 + 2E^2 - 2/3)(e_\odot{}^2 + 2/3)(e^2 + 2/3)- \\ \tfrac{25}{9}C^2(e_\odot{}^2 - 2/5)e^2\cos(2\theta_3)) - n_\odot(P_1 - P_3) \end{aligned} \tag{10}$$

Since P_1 and P_2 are constants, the term additive $-n_\odot P_1$ is neglected. So our final Hamiltonian in the terms of (P_i, Θ_i) variables, is a one degree of freedom, conservative Hamiltonian.

$$\begin{aligned} R2_{res} = \tfrac{27\mu' {n_\odot}^2 a^2}{16}((A^2 + B^2 + C^2 + D^2 + 2E^2 - 2/3)({e_\odot}^2 + 2/3)(e^2 + 2/3) - \\ \tfrac{25}{9} C^2 ({e_\odot}^2 - 2/5) e^2 \cos(2\theta_3)) + n_\odot P_3 \end{aligned} \tag{11}$$

Rewriting P_3 as a function of the orbital elements and adding the perturbation due to the oblateness of the planet (J_2), we obtain the resonant disturbing potential (evection resonance) given by equation (12), where we included the dominant part of the oblateness:

$$\begin{aligned} R2_{res} = -\tfrac{1}{4} \tfrac{J_2 {R_p}^2 n^2 (-2+3(\sin(i))^2)}{(-e^2+1)^{3/2}} + \tfrac{27\mu' {n_\odot}^2 a^2}{16}((A^2 + B^2 + C^2 + D^2 + \\ 2E^2 - 2/3)({e_\odot}^2 + 2/3)(e^2 + 2/3) - \tfrac{25}{9} C^2 ({e_\odot}^2 - 2/5) e^2 \cos(2\theta_3)) + \\ n_\odot \sqrt{\mu a}(\sqrt{1 - e^2} - 1) \end{aligned} \tag{12}$$

we emphasize that this Hamiltonian is only a very crude model, designed to obtain a first idea of the phase space of the evection resonance, for very close satellite. Even an inclusion of a single C_{22} term, modifies drastically the present model since the problem becomes, at least a two degree of freedom system, usually non integrable. This more general Hamiltonian can be studied in a next work.

3 Calculating the resonant semimajor axis

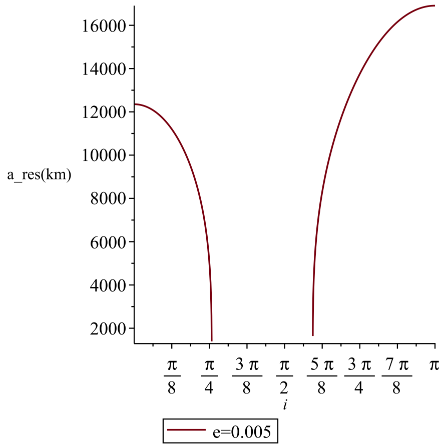
In this section, we calculate the resonant semimajor axis of some celestial bodies of the solar system using equations (4) and (7). In some cases, we find the resonant semimajor axis due to the evection resonance smaller than the reference radius of the central body. In this case, we say that there is no resonant semimajor axis due to evection resonance. The model considering the C_{22} term to calculate the resonant semimajor axis due to evection resonance, can be considered for bodies that are in a spin-orbit resonance. These bodies have significant equatorial ellipticity (C_{22}) (for example, Moon, Haumea and some satellites of the solar system), because it is forced by the gravitational tidal torque of the disturbing body. Hence the classic model is used, which takes into account only the J_2 term. Using equation (12) we obtain the level curves for the case where we find the resonant semimajor axis greater than the reference radius of the central body.

I) Celestial body of the solar system: Earth

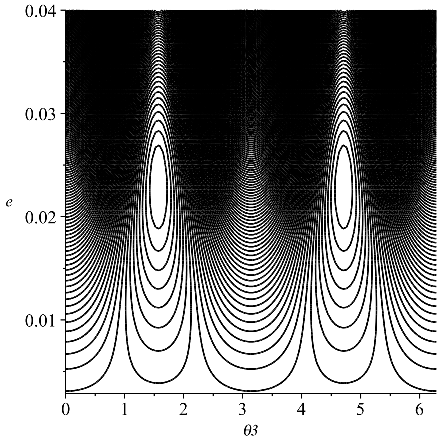
Using equation (4) we show the variation of the semimajor axis resonant of the Earth with respect to inclination, as we can see in Figure 1(a). To compare with [8] take the inclination of $i = 1°$ and $e = 0.005$ to calculate the resonant semimajor axis. We found $a_{res} = 12,350.58474$ km, which is in agreement with [8]. Note that when $i = 180°$ we have $a_{res} = 16,907.65931$ km. Now, using equation (12) we can construct the level curve of the eccentricity versus the resonant angle, which is also in agreement with [8] as we can see in Figure 1(b) of this work and Figure (2.2) of [8]. Note that in Figure 1(b) appear the regions that librates around the equilibrium point.

II) Celestial body of the solar system: Mercury

In the case of Mercury, there is no resonant semimajor axis (using equation (4)), since the value found is smaller than the reference radius of the planet, as shown in Figure 2(a). This is in agreement with [5], where the authors comment that all values of eccentricity of the orbits affected by the resonance lead to pericenters that are inside the physical radius of the planet. As in the case of Mercury the C_{22} term is of the same order of magnitude J_2 (see [13]), then we developed an equation for the resonant semimajor axis taking into account the J_2 and C_{22} terms. Equation (7) represents this model. In this equation we stall have two additional variables, inclination



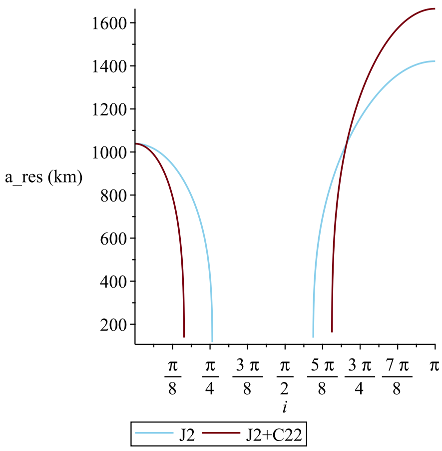
(a)



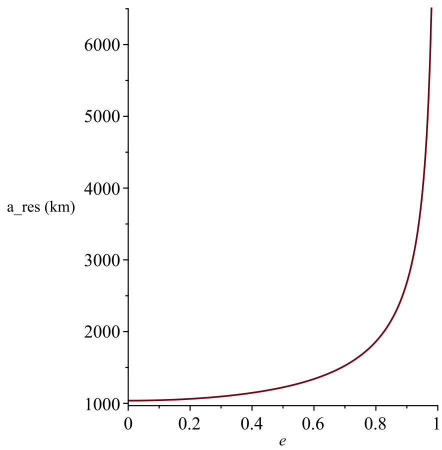
(b)

Figura 1: Initial Condition: $e = 0.005$. celestial body considered: Earth. (a) Resonant semimajor axis (evection) versus inclination, using equation (4). (b) Level curves of the equation (12) showing the variation of the eccentricity as a function of the resonant angle. Initial Condition: $a_{res} =$ 12350.58474 km, $i = 1°$.

and longitude of the ascending node. Again there is no resonant semimajor axis for the case of Mercury, even considering the C_{22} term, as shown in Figure 2(a). That is, we verified that the evection resonance does not contribute to the dynamics of an artificial satellite around Mercury. We also present Figure 2(b), eccentricity versus semimajor axis resonant, note that the possible values of the semimajor axis are valid for eccentricities above 0.8, which is impractical for a satellite near Mercury.



(a)

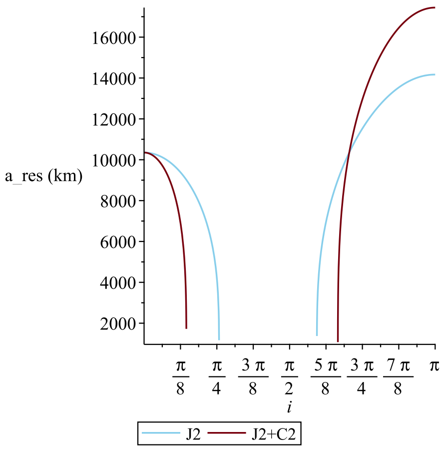


(b)

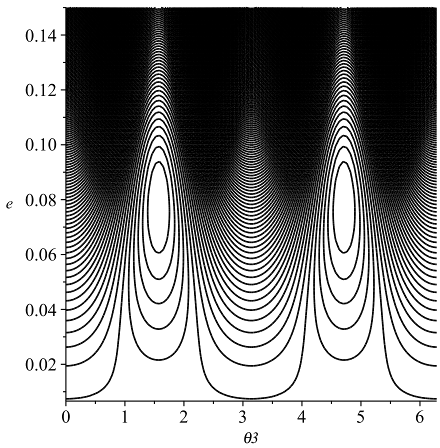
Figura 2: Initial Condition: $e = 0.01$. Celestial body considered: Mercury. (a) Resonant semimajor axis (evection) versus inclination. (b) Resonant semimajor axis (evection) versus eccentricity, using equation (4).

III) Celestial body of the solar system: Dwarf planet Haumea

It is known that Haumea is a triaxial ellipsoid with principal semi-axes $A = 1161$ km $> B = 852$ km $> C = 513$ km (see [10]). Haumea does not have the well-defined spherical harmonic values. Here, we calculate the values of J_2 and C_{22} using the formulas presented in [10]. But, in [2], a review of the calculation of the spherical harmonics (J_2, J_4 and C_{22}) of Haumea is presented considering different equations found in the literature and the result is compared with other authors. We find $J_2 = 0.76$ and $C_{22} = 0.61$, this value is in accordance with the Table 1 given in reference [10]. Figure 3(a) shows the behavior of the resonant semimajor axis for several inclination values. Using Equation (4), fixing $i = 1°$ and $e = 0.01$, we find $a = 10,349.50329$ km. Now, fixing $i = 180°$ and $e = 0.01$, we find $a = 14,168.22598$ km. In the case of Haumea, the C_{22} term must also be taken into account because of its order of magnitude when compared to the J_2 term (see, for example, [2]). Thus, we also use equation (7), to consider the C_{22} term in the dynamics. According to Figure 3(a), we notice that the minimum and maximum points have been changed to the effect of the C_{22} term. The inclination values have also been changed. Considering equation (12), we constructed the contour of the eccentricity versus the resonant angle. Note that Figure 3(b) shows the regions that librates around the equilibrium point. This figure shows that there is symmetry for resonant angle around 90 and 270 degrees. Other orbits circulate, in this case the eccentricity is excited as shown in Figure 3(b).



(a)



(b)

Figura 3: Initial Condition: $e = 0.01$. celestial body considered: Haumea. (a) Resonant semimajor axis (evection) versus inclination. (b) Level curves of the equation (12). Initial Condition: $a_{res} = 10349.50329$ km, $i = 1°$.

4 Conclusions

In this case when the frequencies of longitude of pericenter of the satellite and the mean motion of the sun are in ratio 1:1, the angle $\varpi - \lambda_{\odot}$ may librate. In principle, the study of this resonance has been done following standard procedure. The main effect is the increase of the eccentricity of the satellite, which can cause the escape of the satellite. Recently, due to the increase of mission exploring asteroids, dwarf planets, satellites of giant planets, etc, it appeared the interest in studying orbiters of these triaxial bodies. This time, contrary to the classical case, the resonant semimajor axis (a_{crit}) is not a unique isolated value. In principle, it is a curve in the plane

(a, h), where h is the longitude of the node of the satellite. Here we just presented some crude aproximations showing the equation that defines a_{crit} for some especial celestial bodies. We intend to study evection resonance with more details in the future.

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