Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 9, n. 1, 2022.

Trabalho apresentado no XLI CNMAC, Unicamp - Campinas - SP, 2022.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics
Preprint

Instability of Differential Equations with Piecewise Constant Argument of Generalized Type

Iguer Luis Domini dos Santos[1]
Departamento de Matemática/UNESP, Ilha Solteira, SP

Abstract. The article establishes a result of Lyapunov instability to differential equations with piecewise constant argument of generalized type (EPCAG), through the qualitative study of solutions for EPCAG via functions of continuous time. Using the result established in the article, we study the instability of a logistic equation with piecewise constant argument of generalized type.

Keywords. Nonlinear Differential Equations, Piecewise Constant Argument of Generalized Type, Lyapunov Stability, Instability

1 Introduction and Preliminary

The present article studies differential equations with piecewise constant argument of generalized type (EPCAG). The stability study for such differential equations can be found, for example, in [2–5], [7] and [10, 11]. It can be seen that instability is treated in [5] and [10]. In [5] the instability is treated for a logistic equation EPCAG by reducing the equation into a difference equation. On the other hand, in [10] the instability is treated for a scalar differential-difference equation.

Suppose $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}^+$ denotes the set of nonnegative real numbers, that is, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{R}^+ = [0, \infty)$. In addition, we will denote the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$, by $\|.\|$. Consider a sequence $\{\theta_i\}_{i\in\mathbb{N}}$ of real numbers such that $0 = \theta_0 < \theta_1 < \cdots < \theta_i < \cdots$ and $\theta_i \to \infty$ as $i \to \infty$.

We study here the following class of nonlinear differential equations with piecewise constant arguments

$$\dot{x}(t) = f(t, x(t), x(\beta(t))) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $t \in \mathbb{R}^+$ and $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$. Here $\dot{x}(t)$ denotes the derivative of function $x : \mathbb{R}^+ \to \mathbb{R}^n$ at t. Eq. (1) was considered in [3] for the investigation of stability with the second Lyapunov method. To the best of our knowledge, the literature lacks instability results via functions of continuous time.

Based on Lyapunov's first instability theorem (see [12]), the present work establishes an instability result for EPCAG. Then we use the result established here to study the instability of a class of EPCAG that determines the logistic equation with piecewise constant argument of generalized type (see [5]).

A continuous function $x(\cdot)$ is a solution to Eq. (1) on $\mathbb{R}^+$ if it satisfies the Eq. (1) on the intervals $[\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$, and the derivative $\dot{x}(t)$ exists everywhere with the possible exception of the points θ_i, $i \in \mathbb{N}$, where one-sided derivatives exist. Let $S(x_0)$ denote the set of solutions $x(\cdot)$ to Eq. (1) on $[0, \infty)$ with $x(0) = x_0$. If $x \in S(x_0)$, we will also use the notation $x(t, 0, x_0)$ to denote $x(t)$. The existence of solutions to EPCAG can be found in [1] and [3].

[1]iguer.santos@unesp.br

As an example of Eq. (1), below we determine the solutions $y(\cdot)$ of the following equation

$$\dot{y}(t) = -by(\beta(t))\big(y(t) + \frac{a}{b}\big), \tag{2}$$

where a and b are nonzero constants of the same sign. If $y(0) = y_0$ and $y_0 + \frac{a}{b} > 0$, suppose that $y(t) + \frac{a}{b} > 0$ for $t \in (0, c_1)$, with $0 < c_1 \leq \theta_1$. Using separation of variables (see [9]),

$$y(t) = -\frac{a}{b} + \big(y_0 + \frac{a}{b}\big)e^{-by_0 t},$$

and we may conclude that $c_1 = \theta_1$. Since $y \in S(y_0)$ is continuous, $y(\theta_1) = -\frac{a}{b} + \big(y_0 + \frac{a}{b}\big)e^{-by_0\theta_1} = y_1$. If $y(t) + \frac{a}{b} > 0$ for $t \in (\theta_1, c_2)$, with $\theta_1 < c_2 \leq \theta_2$, we have

$$y(t) = -\frac{a}{b} + \big(y_1 + \frac{a}{b}\big)e^{-by_1(t-\theta_1)},$$

and then $c_2 = \theta_2$. By mathematical induction,

$$y(t) = -\frac{a}{b} + \big(y_i + \frac{a}{b}\big)e^{-by_i(t-\theta_i)},$$

for $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$, with $y_i = y(\theta_i)$. Similarly, if $y_0 + \frac{a}{b} < 0$, then $S(y_0) = \{y\}$, where

$$y(t) = -\frac{a}{b} + \big(y_i + \frac{a}{b}\big)e^{-by_i(t-\theta_i)},$$

for $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$, with $y_i = y(\theta_i)$. Now suppose that $y_0 = -\frac{a}{b}$ and $t \in [\theta_0, \theta_1)$. Then

$$\dot{y}(t) = ay(t) + \frac{a^2}{b},$$

and from variation of constants formula (see [6]) we may conclude that

$$y(t) = e^{ta}y_0 + \int_0^t e^{(t-s)a}\frac{a^2}{b}ds,$$

that is, $y(t) = -\frac{a}{b}$ for $t \in [\theta_0, \theta_1)$. Since $y \in S(-\frac{a}{b})$ is continuous, $y(\theta_1) = y_1 = -\frac{a}{b}$. Hence

$$\dot{y}(t) = ay(t) + \frac{a^2}{b},$$

for $t \in [\theta_1, \theta_2)$. From variation of constants formula,

$$y(t) = e^{(t-\theta_1)a}y_1 + \int_{\theta_1}^t e^{(t-s)a}\frac{a^2}{b}ds,$$

and then $y(t) = -\frac{a}{b}$ for $t \in [\theta_1, \theta_2)$. By mathematical induction $y(t) = -\frac{a}{b}$ for each $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$. We conclude that $S(-\frac{a}{b}) = \{y\}$, where $y(t) = -\frac{a}{b}$ for $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$.

We note that the equilibria of Eq. (2) are given by $y = 0$ and $y = -\frac{a}{b}$.

Throughout the article, we assume the following hypothesis.

(H) For each $x \in \mathbb{R}^n$, the function $(t, y) \mapsto f(t, x, y)$ is continuous.

Thus, if $x \in S(x_0)$ then $\dot{x}(\cdot)$ is continuous as a function defined in $[\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$.

DOI: 10.5540/03.2022.009.01.0264

2 Absolutely Continuous Functions

In this section are considered concepts and results that will be used in the development of the main result (Theorem 3.1). Below we have the definition of an absolutely continuous function (see for instance [13] and [15]).

Definition 2.1. *A function $x : [a, b] \to \mathbb{R}^n$ is called absolutely continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any countable collection of disjoint subintervals $[a_k, b_k]$ of $[a, b]$ satisfying*

$$\sum (b_k - a_k) < \delta,$$

we have

$$\sum |x(b_k) - x(a_k)| < \varepsilon.$$

We can also define an absolutely continuous function on a given interval $I \subset \mathbb{R}$. We have the following result concerning absolutely continuous functions. Here we consider the notion of Lebesgue integral.

Theorem 2.1 ([13])**.** *In order that the function $F(x)$ be an indefinite integral, it is necessary and sufficient that it be absolutely continuous.*

As we can see in [15], an absolutely continuous function $x : [a, b] \to \mathbb{R}^n$ is differentiable almost everywhere, and its derivative $\dot{x}(\cdot)$ is a Lebesgue integrable function. We also note the Newton-Leibniz formula is true; that is,

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} \dot{x}(s)ds$$

for all $t_1, t_2 \in [a, b]$, $t_1 < t_2$.

Let $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function and let $x \in S(x_0)$. Then the function $w : \mathbb{R}^+ \to \mathbb{R}$ defined by $w(t) = V(t, x(t))$ is absolutely continuous. Indeed, since

$$x(t) = x(\theta_i) + \int_{\theta_i}^{t} \dot{x}(s)ds$$

for $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$, from Theorem 2.1 the function x is absolutely continuous on $[\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$. Since x is a function continuous on $\mathbb{R}^+$, we conclude that x is absolutely continuous on $[\theta_i, \theta_{i+1}]$, $i \in \mathbb{N}$. Then w is absolutely continuous on $[\theta_i, \theta_{i+1}]$, $i \in \mathbb{N}$, because V is locally Lipschitz continuous. Thus, w is absolutely continuous.

3 Lyapunov Instability

From Lyapunov's first instability theorem and its proof (as [12, Theorem 9.16]), in Theorem 3.1 we establish an instability result to system given by Eq. (1). Theorem 3.1 is formulated by using functions of continuous time V, and its proof is similar to [12, Theorem 9.16]. The functions of continuous time V in Theorem 3.1 are analogous to Lyapunov functions for systems of ordinary differential equations (see for instance [8] and [16]).

Concepts of Lyapunov stability for the system given by Eq. (1) are formulated in a similar way to ordinary differential equations (see [12] and [14]). Below, we consider concepts of stability (in the sense of Lyapunov) to solution $x \equiv 0$ of Eq. (1) with initial condition $x(0) = x_0$. For this, we can assume that $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}^+$.

Definition 3.1. *The equilibrium $x = 0$ is stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that if $\|x_0\| < \delta$, then $\|x(t, 0, x_0)\| < \varepsilon$ for all $t \geq 0$.*

Definition 3.2. *The equilibrium $x = 0$ of Eq. (1) is unstable if it is not stable.*

Definition 3.3. *We say that a continuous function $\psi : [0, r_1] \to \mathbb{R}^+$ (respectively, $\psi : [0, \infty) \to \mathbb{R}^+$) belongs to class $\mathcal{K}$ ($\psi \in \mathcal{K}$), if $\psi(0) = 0$ and if ψ is strictly increasing on $[0, r_1]$ (respectively, on $[0, \infty)$).*

Below, $B(h)$ denotes the open ball of radius h centered at origin,

$$B(h) = \{x \in \mathbb{R}^n : \|x\| < h\}.$$

Theorem 3.1. *Let $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Suppose that in every neighborhood of the origin there are points x such that $V(0, x) > 0$. Also suppose that there exist functions $\psi_1, \psi_2 \in \mathcal{K}$ satisfying the following conditions:*

(i) for some $h > 0$, $|V(t, x)| \leq \psi_1(\|x\|)$ for all $(t, x) \in \mathbb{R}^+ \times B(h)$;

(ii) $\frac{d}{dt}V(t, \phi(t)) \geq \psi_2(\|\phi(t)\|)$ for $t \in (\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$, and for all $\phi \in S(x_0)$ with $\|\phi(t)\| < h$, for each $t \in \mathbb{R}^+$.

Then the equilibrium $x = 0$ of Eq. (1) is unstable.

Proof. Take $\varepsilon > 0$ such that $\varepsilon \leq h$. Consider a sequence of points $\{x_m\}_{m\in\mathbb{N}}$ satisfying $0 < \|x_m\| < \varepsilon$, $V(0, x_m) > 0$ and

$$\lim_{m\to\infty} x_m = 0.$$

Let $\phi_m \in S(x_m)$ and $w_m : \mathbb{R}^+ \to \mathbb{R}$, with $w_m(t) = V(t, \phi_m(t))$. Since w_m is absolutely continuous,

$$w_m(t) - w_m(0) = \int_0^t \frac{d}{ds} w_m(s) ds$$

for all $t \in \mathbb{R}^+$. Then $\|\phi_m(t_m)\| = \varepsilon$ for some $t_m \in \mathbb{R}^+$. For otherwise,

$$\begin{aligned}
\psi_1(\|\phi_m(t)\|) \geq V(t, \phi_m(t)) &= w_m(t) \\
&= w_m(0) + \int_0^t \frac{d}{ds} w_m(s) ds \\
&= w_m(0) + \int_0^t \frac{d}{ds} V(s, \phi_m(s)) ds \\
&\geq w_m(0) + \int_0^t \psi_2(\|\phi_m(s)\|) ds \\
&\geq w_m(0)
\end{aligned}$$

for all $t \in \mathbb{R}^+$. Hence

$$\|\phi_m(t)\| \geq \psi_1^{-1}(w_m(0)) = \alpha_m > 0$$

for all $t \in \mathbb{R}^+$. Thus,

$$\begin{aligned}
\psi_1(\varepsilon) > \psi_1(\|\phi_m(t)\|) &\geq w_m(t) \\
&\geq w_m(0) + \int_0^t \psi_2(\|\phi_m(s)\|) ds \\
&\geq w_m(0) + \int_0^t \psi_2(\alpha_m) ds \\
&= w_m(0) + t\psi_2(\alpha_m)
\end{aligned}$$

for all $t \in \mathbb{R}^+$. Taking $t \to \infty$ we get a contradiction. So, the equilibrium $x = 0$ is unstable. □

As an example, for the use of Theorem 3.1, we consider a class of EPCAG determined by a logistic equation given in [5]. So, consider the logistic equation

$$\dot{x}(t) = \big(a - bx(\beta(t))\big)x(t), \tag{3}$$

where a and b are nonzero constants of the same sign, $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$. If $a > 0$, using Theorem 3.1 we can conclude that the equilibrium $x = 0$ is unstable. Let $h > 0$ be such that $hb < a$. Let $V(t, x) = x^2$ and $\phi \in S(x_0)$. If $\|\phi(t)\| < h$ for all $t \in \mathbb{R}^+$, we have

$$\begin{aligned} \frac{d}{dt}V(t,\phi(t)) &= 2\phi(t)\frac{d}{dt}\phi(t) \\ &= 2\big(a - b\phi(\beta(t))\big)(\phi(t))^2 \\ &\geq 2\big(a - bh\big)(\phi(t))^2 \\ &= \psi_2(\|\phi(t)\|) \end{aligned}$$

for $t \in (\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$, where $\psi_2(u) = 2\big(a - bh\big)u^2$. From Theorem 3.1 the equilibrium $x = 0$ of Eq. (3) is unstable.

Using the transformation $y = x - \frac{a}{b}$, Eq. (3) becomes Eq. (2). The qualitative behavior of the equilibrium $x = \frac{a}{b}$ (of Eq. (3)) is equivalent to that of the equilibrium $y = 0$ (of Eq. (2)). Finding a function V as in Theorem 3.1 may not be an easy task. Similar to the search for a Lyapunov function for systems of ordinary differential equations (see [8]).

4 Final Considerations

The article establishes a result on Lyapunov instability for EPCAG from Lyapunov's second method for ordinary differential equations. The result stablished here is an analogy of Lyapunov's first instability theorem. As future work, there is the possibility of further analogies between EPCAG and instability results for ordinary differential equations, such as Lyapunov's second instability theorem and Chetaev's instability theorem.

References

[1] M. U. Akhmet. "Integral manifolds of differential equations with piecewise constant argument of generalized type". In: **Nonlinear Anal.** 66.2 (2007), pp. 367–383. DOI: 10.1016/j.na.2005.11.032.

[2] M. U. Akhmet. "Stability of differential equations with piecewise constant arguments of generalized type". In: **Nonlinear Anal.** 68.4 (2008), pp. 794–803. DOI: 10.1016/j.na.2006.11.037.

[3] M. U. Akhmet, D. Aruğaslan, and Yılmaz E. "Method of Lyapunov functions for differential equations with piecewise constant delay". In: **J. Comput. Appl. Math.** 235.16 (2011), pp. 4554–4560. DOI: 10.1016/j.cam.2010.02.043.

[4] M. S. Alwan, X. Liu, and W. C. Xie. "Comparison principle and stability of differential equations with piecewise constant arguments". In: **J. Franklin Inst.** 350.2 (2013), pp. 211–230. DOI: 10.1016/j.jfranklin.2012.08.016.

[5] D. Aruğaslan and L. Güzel. "Stability of the logistic population model with generalized piecewise constant delays". In: **Adv. Difference Equ.** 2015.173 (2015), p. 10. DOI: 10.1186/s13662-015-0521-8.

[6] Earl A. Coddington and Norman Levinson. **Theory of ordinary differential equations**. New York: McGraw-Hill, 1955.

[7] K. Gopalsamy and P. Liu. "Persistence and global stability in a population model". In: **J. Math. Anal. Appl.** 224.1 (1998), pp. 59–80. DOI: 10.1006/jmaa.1998.5984.

[8] W. Hahn. **Stability of motion**. Berlin: Springer, 1967.

[9] E. L. Ince. **Ordinary differential equations**. New York: Dover, 1956.

[10] A. F. Ivanov. "Global dynamics of a differential equation with piecewise constant argument". In: **Nonlinear Anal.** 71.12 (2009), e2384–e2389. DOI: 10.1016/j.na.2009.05.030.

[11] H. Li, Y. Muroya, and R. Yuan. "A sufficient condition for global asymptotic stability of a class of logistic equations with piecewise constant delay". In: **Nonlinear Anal. Real World Appl.** 10.1 (2009), pp. 244–253. DOI: 10.1016/j.nonrwa.2007.09.006.

[12] Richard K. Miller and Anthony N. Michel. **Ordinary differential equations**. New York: Academic Press, 1982.

[13] F. Riesz and B. Sz.-Nagy. **Functional analysis**. New York: Frederick Ungar Publishing Co., 1955.

[14] T. C. Sideris. **Ordinary differential equations and dynamical systems**. Paris: Atlantis Press, 2013.

[15] G. V. Smirnov. **Introduction to the theory of differential inclusions**. Providence: American Mathematical Society, 2002.

[16] T. Yoshizawa. **Stability theory by Liapunov's second method**. Tokyo: The Mathematical Society of Japan, 1966.

