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Second-Order KKT-Invexity in Continuous-Time Optimization

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Abstract. In this paper, we adapt to the context of continuous-time optimization a concept of generalized convexity adequate to work with second-order stationary solutions, which are solutions that satisfy second-order necessary optimality conditions. We show that the second-order necessary optimality conditions become sufficient when the problem satisfies such a generalized convexity concept. We also show that, under a certain regularity assumption, this concept is as general as possible, in the sense that if the problem is such that every second-order stationary solution is an optimal solution, then the problem necessarily satisfies the generalized convexity concept.

Palavras-chave. Continuous-Time Optimization, Sufficient Optimality Conditions, Second-Order KKT-Invexity

1 Introduction

In this work, we focus on second-order sufficient optimality conditions: we show that the second-order necessary optimality conditions become sufficient when the continuous-time optimization problem satisfies a certain concept of generalized convexity. Such a concept, introduced by Ivanov [3] for classical nonlinear optimization problems (in finite dimensions), is called second-order KKT-invexity. It is derived from the concept of invexity, which was presented earlier by Hanson in [2] to generalize a desirable property of convex problems: every stationary point is a global minimizer. Later, Craven and Glover [1] showed that a function is invex if, and only if, every stationary point is a global minimizer. However, although every stationary point is a global minimizer in invex mathematical programming problems, there are noninvex problems where this property holds. So, Martin [4] proposed the concept of KKT-invexity. While the property of every stationary point being a global minimizer was maintained for KKT-invex problems, Martin showed that the converse is true. That is, every stationary point is a global minimizer if, and only if, the problem is KKT-invex. In the same vein, Ivanov [3] proposes second-order KKT-invexity for nonlinear programming problems. His main result says that every second-order stationary point is a global minimizer if, and only if, the problem is second-order KKT-invex. In this work, we adapt Ivanov's definition to the continuous-time context and reach a similar result. However, to show that problems in which every second-order stationary solution is an optimal solution are necessarily second-order KKT-invex, we impose a regularity condition, namely the linear independence constraint qualification. Ivanov gets his result without any additional assumptions. Besides, he uses linear programming results in his demonstration, which we also do. Notwithstanding, in the continuous-time context, regularity conditions are needed even in the linear case. Sufficient conditions for continuous-time optimization problems via generalized convexity can be found in the literature, to cite but a few, in de Oliveira and Rojas-Medar [7, 8], Nobakhtian and Pouryayevali [5] and Rojas-Medar et al. [9]. But in these works, second-order stationary solutions are not considered.

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2 Preliminaries

We deal with the continuous-time optimization problem posed as follows:

$$\begin{array}{ll} \text{minimize} & F(z) = \int_0^T f(z(t),t)dt \\ \text{subject to} & g(z(t),t) \leq 0 \text{ a.e. in } [0,T], \\ & z \in L^\infty([0,T];\mathbb{R}^n), \end{array} \tag{CTP}$$

where $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ and $g : \mathbb{R}^n \times [0,T] \to \mathbb{R}^m$. The integral is in the Lebesgue sense. The set of all feasible solutions is denoted by Ω, i.e.,

$$\Omega = \{z \in L^\infty([0,T];\mathbb{R}^n) : g(z(t),t) \leq 0 \text{ a.e. in } [0,T]\}.$$

We denote $I = \{1,\dots,m\}$ and, given $\bar{z} \in \Omega$, $I_a(t) = \{i \in I : g_i(\bar{z}(t),t) = 0\}$. A feasible solution $\bar{z} \in \Omega$ is said to be an optimal solution of (CTP) if $F(\bar{z}) \leq F(z)$ for all $z \in \Omega$. We say that the basic hypotheses are valid in $\bar{z} \in \Omega$ if there exists $\varepsilon > 0$ such that

(H1) $f(z,\cdot)$ and $g(z,\cdot)$ are measurable for each z;

$f(\cdot,t)$ and $g(\cdot,t)$ are twice continuously differentiable in $\bar{z}(t) + \varepsilon\bar{B}$ a.e. in $[0,T]$;[2]

$g(z(\cdot),\cdot)$ is essentially bounded in $[0,T]$ for each $z \in \Omega$ such that $\|z - \bar{z}\|_\infty < \varepsilon$;

(H2) there exist $K_f > 0$ and $K_g > 0$ such that

$$|\nabla f(\bar{z}(t),t)| + |\nabla^2 f(\bar{z}(t),t)| \leq K_f \text{ and } |\nabla g(\bar{z}(t),t)| + |\nabla^2 g(\bar{z}(t),t)| \leq K_g \text{ a.e. in } [0,T].$$

We assume throughout the paper that (H1) and (H2) are valid. Now, we define critical directions and second-order stationary solutions for (CTP).

Definition 2.1. *Let $\bar{z} \in \Omega$. We define $\mathscr{D}(\bar{z})$ as the set of all directions $\gamma \in L^\infty([0,T];\mathbb{R}^n)$ such that*

$$\nabla f(\bar{z}(t),t)^\top \gamma(t) \leq 0 \textit{ and } \nabla g_i(\bar{z}(t),t)^\top \gamma(t) \leq 0,\ i \in I_a(t),\ \textit{a.e. in } [0,T].$$

Definition 2.2. *A feasible solution $\bar{z} \in \Omega$ is said to be a second-order stationary solution of (CTP) if given $\gamma \in \mathscr{D}(\bar{z})$, there exists $u \in L^\infty([0,T];\mathbb{R}^p)$ such that*

$$\nabla f(\bar{z}(t),t) + \sum_{i \in I} u_i(t) \nabla g_i(\bar{z}(t),t) = 0 \textit{ a.e. in } [0,T],$$
$$u_i(t) \geq 0,\ u_i(t) g_i(\bar{z}(t),t) = 0 \textit{ a.e. in } [0,T],\ i \in I,$$

and

$$\int_0^T \gamma(t)^\top \Big[\nabla^2 f(\bar{z}(t),t) + \sum_{i \in I} u_i(t) \nabla^2 g_i(\bar{z}(t),t)\Big]\gamma(t)dt \geq 0.$$

The following is the definition of the concept of generalized convexity.

Definition 2.3. *We say that (CTP) is second-order KKT-invex at $\bar{z} \in \Omega$ if for all $z \in \Omega$ there exit $\eta = \eta(z,\bar{z}) \in L^\infty([0,T];\mathbb{R}^n)$, $\omega = \omega(z,\bar{z}) \in L^\infty([0,T];\mathbb{R})$ and $\gamma = \gamma(z,\bar{z}) \in L^\infty([0,T];\mathbb{R}^n)$ such that*

$$F(z) - F(\bar{z}) \geq \int_0^T [\nabla f(\bar{z}(t),t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 f(\bar{z}(t),t)\gamma(t)]dt,$$
$$0 \geq \nabla g_i(\bar{z}(t),t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 g_i(\bar{z}(t),t)\gamma(t),\ i \in I_a(t),\ \textit{a.e. in } [0,T],$$
$$\omega(t) \geq 0 \textit{ a.e. in } [0,T],\ \gamma \in \mathscr{D}(\bar{z}).$$

[2] B denotes the open unit ball centered at the origin in $\mathbb{R}^n$.

When (CTP) is second-order KKT-invex at every $\bar{z} \in \Omega$, we say that (CTP) is second-order KKT-invex.

By taking $\eta(t) = z(t) - \bar{z}(t)$, $\omega(t) = 1$ and $\gamma(t) = 0$ a.e. in $[0, T]$ in the definition above, we see that every convex problem is a second-order KKT-invex one.

3 Auxiliary Results

To facilitate the demonstration of the main results, we will use some auxiliary results, which we will state in the form of the following three propositions.

Proposition 3.1. *Let $\bar{z} \in \Omega$ e $\gamma \in \mathscr{D}(\bar{z})$. If $\bar{z}$ is a second-order stationary solution of (CTP), then the system*

$$\int_0^T [\nabla f(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 f(\bar{z}(t), t)\gamma(t)]dt < 0, \tag{1}$$

$$\nabla g_i(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 g_i(\bar{z}(t), t)\gamma(t) \leq 0, \ i \in I_a(t), \ \text{a.e. in } [0, T], \tag{2}$$

has no solution $(\eta, \omega) \in L^\infty([0, T]; \mathbb{R}^n) \times L^\infty([0, T]; \mathbb{R}_+)$.

Proof. On the contrary, assume that system (1)-(2) has a solution $(\eta, \omega) \in L^\infty([0, T]; \mathbb{R}^n) \times L^\infty([0, T]; \mathbb{R}_+)$. Let $\tilde{\gamma}(t) = \sqrt{\omega(t)}\gamma(t)$ a.e. in $[0, T]$. Since $\bar{z} \in \Omega$ is a second-order stationary solution, given $\tilde{\gamma} \in \mathscr{D}(\bar{z})$, there exists $u \in L^\infty([0, T]; \mathbb{R}^p)$ such that

$$\int_0^T \Big[\nabla f(\bar{z}(t), t) + \sum_{i \in I_a(t)} u_i(t) \nabla g_i(\bar{z}(t), t)\Big]^\top \eta(t) dt$$
$$+ \int_0^T \omega(t)\gamma(t)^\top \Big[\nabla^2 f(\bar{z}(t), t) + \sum_{i \in I_a(t)} u_i(t) \nabla^2 g_i(\bar{z}(t), t)\Big] \gamma(t) dt \geq 0. \tag{3}$$

On the other hand, by multiplying (2) by $u_i(t)$, summing over $I_a(t)$ and integrating in $[0, T]$, and, at last, summing with (1), we obtain

$$\int_0^T [\nabla f(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 f(\bar{z}(t), t)\gamma(t)]dt$$
$$+ \int_0^T \sum_{i \in I_a(t)} u_i(t)[\nabla g_i(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 g_i(\bar{z}(t), t)\gamma(t)]dt < 0.$$

By rearranging the terms we get to a contradiction to (3). ☐

Proposition 3.2. *Let $\bar{z} \in \Omega$ and $\gamma \in \mathscr{D}(\bar{z})$. Assume that there exists $\tilde{K} > 0$ such that*

$$\det(\nabla g^{I_a(t)}(\bar{z}(t), t) \nabla g^{I_a(t)}(\bar{z}(t), t)^\top) \geq \tilde{K} \ \text{a.e. in } [0, T]. \tag{4}$$

If system (1)-(2) does not have any solution $(\eta, \omega) \in L^\infty([0, T]; \mathbb{R}^n) \times L^\infty([0, T]; \mathbb{R}_+)$, then $\bar{z}$ is a second-order stationary solution of (CTP).

Proof. Let us consider the following continuous-time linear programming problem:

$$\begin{array}{ll} \min & \Phi(\eta, \omega) = \int_0^T [\nabla f(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 f(\bar{z}(t), t)\gamma(t)]dt \\ \text{s.t} & \delta_i(t)[\nabla g_i(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 g_i(\bar{z}(t), t)\gamma(t)] \leq 0 \text{ a.e. in } [0, T], \ i \in I, \\ & -\omega(t) \leq 0 \text{ a.e. in } [0, T], \ (\eta, \omega) \in L^\infty([0, T]; \mathbb{R}^n \times \mathbb{R}), \end{array} \tag{AP}$$

where $\delta_i(t) = 1$ for $i \in I_a(t)$ and $\delta_i(t) = 0$ otherwise. Clearly, $(\bar{\eta}, \bar{\omega}) = (0, 0)$ is feasible and $\Phi(\bar{\eta}, \bar{\omega}) = 0$. Since $\Phi(\eta, \omega) \geq 0$ for all feasible solution (η, ω), it follows that $(\bar{\eta}, \bar{\omega})$ is an optimal solution of (AP). Provided (4) holds, we see that the regularity condition from Theorem 2.2 in de Oliveira [6] is valid. It follows that there exists $(u, v) \in L^\infty([0, T]; \mathbb{R}^m) \times L^\infty([0, T]; \mathbb{R})$ such that, for almost all $t \in [0, T]$, $u(t) \geq 0$, $v(t) \geq 0$ and

$$\begin{bmatrix} \nabla f(\bar{z}(t), t) \\ \gamma(t)^\top \nabla^2 f(\bar{z}(t), t)\gamma(t) \end{bmatrix} + \sum_{i \in I} u_i(t)\delta_i(t) \begin{bmatrix} \nabla g_i(\bar{z}(t), t) \\ \gamma(t)^\top \nabla^2 g_i(\bar{z}(t), t)\gamma(t) \end{bmatrix} + v(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By defining $\bar{u}_i(t) = u_i(t)\delta_i(t)$ a.e. in $[0, T]$, we obtain

$$\nabla f(\bar{z}(t), t) + \sum_{i \in I} \bar{u}_i(t) \nabla g_i(\bar{z}(t), t) = 0 \text{ a.e. in } [0, T],$$
$$\bar{u}_i(t) \geq 0, \ \bar{u}_i(t) g_i(\bar{z}(t), t) = 0 \text{ a.e. in } [0, T], \ i \in I,$$

and

$$\int_0^T \gamma(t)^\top \Big[\nabla^2 f(\bar{z}(t), t) + \sum_{i \in I} \bar{u}_i(t) \nabla^2 g_i(\bar{z}(t), t)\Big]\gamma(t) dt = \int_0^T v(t) dt \geq 0.$$

Therefore, $\bar{z}$ is a second-order stationary solution. □

Proposition 3.3. *(CTP) is second-order KKT-invex at $\bar{z} \in \Omega$ if, and only if, for all $z \in \Omega$ with $F(z) < F(\bar{z})$, there exist $\eta = \eta(z, \bar{z}) \in L^\infty([0, T]; \mathbb{R}^n)$, $\omega = \omega(z, \bar{z}) \in L^\infty([0, T]; \mathbb{R})$ and $\gamma = \gamma(z, \bar{z}) \in L^\infty([0, T]; \mathbb{R}^n)$ such that*

$$\int_0^T [\nabla f(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 f(\bar{z}(t), t)\gamma(t)] dt < 0, \tag{5}$$
$$\nabla g_i(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 g_i(\bar{z}(t), t)\gamma(t) \leq 0, \ i \in I_a(t), \ \textit{a.e. in } [0, T], \tag{6}$$
$$\omega(t) \geq 0 \ \textit{a.e. in } [0, T], \ \gamma \in \mathscr{D}(\bar{z}). \tag{7}$$

Proof. If (CTP) is second-order KKT-invex at $\bar{z}$, it is obvious that there exist η, ω and γ satisfying (5)-(7) for all $z \in \Omega$ with $F(z) < F(\bar{z})$.

Assume that for all $z \in \Omega$ with $F(z) < F(\bar{z})$ there exist η, ω and γ that verify (5)-(7). If (CTP) is not second-order KKT-invex at $\bar{z}$, it follows from the definition that there exists $z \in \Omega$ such that system

$$F(z) - F(\bar{z}) \geq \int_0^T [\nabla f(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 f(\bar{z}(t), t)\gamma(t)] dt, \tag{8}$$
$$0 \geq \nabla g_i(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 g_i(\bar{z}(t), t)\gamma(t), \ i \in I_a(t), \text{ a.e. in } [0, T], \tag{9}$$
$$\omega(t) \geq 0 \text{ a.e. in } [0, T], \ \gamma \in \mathscr{D}(\bar{z}), \tag{10}$$

does not have any solution (η, ω, γ). Let us put $(\bar{\eta}(t), \bar{\omega}(t), \bar{\gamma}(t)) = (0, 1, 0)$ a.e. in $[0, T]$. Then it holds (9)-(10). Hence, (8) is violated, so that $F(z) < F(\bar{z})$. It follows, then, from the hypothesis that there exist η, ω and γ that verify (5)-(7). Now, let us set

$$\alpha = \frac{F(z) - F(\bar{z})}{\int_0^T [\nabla f(\bar{z}(t), t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 f(\bar{z}(t), t)\gamma(t)] dt} > 0$$

and $(\tilde{\eta}, \tilde{\omega}, \tilde{\gamma}) = (\alpha\eta, \alpha\omega, \gamma)$. Clearly, $(\tilde{\eta}, \tilde{\omega}, \tilde{\gamma})$ satisfies (9)-(10) and, from the definition of α,

$$F(z) - F(\bar{z}) = \int_0^T [\nabla f(\bar{z}(t), t)^\top \tilde{\eta}(t) + \tilde{\omega}(t)\tilde{\gamma}(t)^\top \nabla^2 f(\bar{z}(t), t)\tilde{\gamma}(t)] dt,$$

which is a contradiction to the fact that (8)-(10) does not have any solution. □

It follows from the last result that if $f(\cdot,t)$ has a negative definite hessian matrix at Ω and $g_i(\cdot,t)$, $i \in I_a(t)$, have negative semi-definite hessian matrices at Ω for almost every $t \in [0,T]$, then (CTP) is second-order KKT-invex. Indeed, it is enough to take, for almost every $t \in [0,T]$, $\eta(t) = 0$, $\omega(t) = 1$ and $\gamma(t) \in \mathscr{D}(\bar{z})$ arbitrary. Following, we have an illustrative example.

Exemplo 3.1. *Problem below is second-order KKT-invex:*

$$\begin{array}{ll} \min & F(z) = \int_0^1 [2z(t) - z(t)^2]dt \\ \text{s.t} & z(t)^2 - 2z(t) \leq 0, \\ & z \in L^\infty([0,1];\mathbb{R}). \end{array}$$

It is easy to see that

$$\Omega = \{z \in L^\infty([0,1];\mathbb{R}) : 0 \leq z(t) \leq 2 \text{ a.e. in } [0,1]\}.$$

Let $\bar{z} \in \Omega$. We denote

$$\begin{array}{rcl} T_a &=& \{t \in [0,1] : \bar{z}(t)^2 - 2\bar{z}(t) = 0\} = \{t \in [0,1] : \bar{z}(t) = 0 \text{ or } \bar{z}(t) = 2\}, \\ T_1 &=& \{t \in [0,1] : 0 < \bar{z}(t) \leq 1\}, \\ T_2 &=& \{t \in [0,1] : 1 < \bar{z}(t) < 2\}. \end{array}$$

Let $z, \bar{z} \in \Omega$ such that $F(z) < F(\bar{z})$. As $F(z) \geq 0$ for all $z \in \Omega$, we have that $F(\bar{z}) > F(z) \geq 0$, so that $F(\bar{z}) > 0$, which implies that $\bar{z}(t)$ is not identically equal to 0 almost always in $[0,1]$ and that $\bar{z}(t)$ is not identically equal to 2 almost always in $[0,1]$. Then, set T_a does not have total measure equal to 1, that is, $T_1 \cup T_2$ has positive measure. Define, for almost every $t \in [0,1]$, $\eta(t) = 0$, $\omega(t) = 1$ and

$$\gamma(t) = \begin{cases} -1 \text{ a.e. in } T_1, \\ 0 \text{ a.e. in } T_a, \\ 1 \text{ a.e. in } T_2. \end{cases}$$

Therefore,

$$\begin{aligned} &\int_0^T [\nabla f(\bar{z}(t),t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 f(\bar{z}(t),t)\gamma(t)]dt \\ &\qquad = \int_0^1 [2\eta(t) - 2\bar{z}(t)\eta(t) - 2\omega(t)\gamma(t)^2]dt = -2\int_0^1 \gamma(t)^2 dt = -2\int_{T_1 \cup T_2} \gamma(t)^2 dt < 0, \\ &\nabla g(\bar{z}(t),t)^\top \eta(t) + \omega(t)\gamma(t)^\top \nabla^2 g(\bar{z}(t),t)\gamma(t) \\ &\qquad = [2\bar{z}(t) - 2]\eta(t) + 2\omega(t)\gamma(t)^2 = 2\gamma(t)^2 = 0 \text{ a.e. in } T_a, \\ &\omega(t) = 1 \geq 0 \text{ a.e. in } [0,1]. \end{aligned}$$

It remains to show that $\gamma \in \mathscr{D}(\bar{z})$. We have that

$$\mathscr{D}(\bar{z}) = \{\gamma \in L^\infty([0,1];\mathbb{R}) : \int_0^1 [2 - 2\bar{z}(t)]\gamma(t)dt \leq 0,\ [2\bar{z}(t) - 2]\gamma(t) \leq 0 \text{ a.e. in } T_a\}.$$

It is clear that $[2\bar{z}(t) - 2]\gamma(t) \leq 0$ a.e. in T_a. Let us notice that $0 \leq 2 - 2\bar{z}(t) < 2$ a.e. in T_1 and $-2 < 2 - 2\bar{z}(t) < 0$ a.e. in T_2. Hence,

$$\begin{array}{rcl} \int_0^1 [2 - 2\bar{z}(t)]\gamma(t)dt &=& \int_{T_a} [2 - 2\bar{z}(t)]\gamma(t)dt + \int_{T_1} [2 - 2\bar{z}(t)]\gamma(t)dt + \int_{T_2} [2 - 2\bar{z}(t)]\gamma(t)dt \\ &=& -\int_{T_1} [2 - 2\bar{z}(t)]dt + \int_{T_2} [2 - 2\bar{z}(t)]dt \leq 0. \end{array}$$

Thus, by Proposition 3.3, the problem is second-order KKT-invex.

4 Main Results

In this section we state and prove our main results. Next, we show that second-order KKT-invexity is a sufficient optimality condition.

Theorem 4.1. *Let $\bar{z} \in \Omega$ be a second-order stationary solution of (CTP). If (CTP) is second-order KKT-invex at $\bar{z}$, then $\bar{z}$ is an optimal solution.*

Proof. If $\bar{z}$ is not an optimal solution, there exists $z \in \Omega$ with $F(z) < F(\bar{z})$. It follows from Proposition 3.3 that there exist η, ω and γ such that (5)-(7) are valid. But this contradicts Proposition 3.1. □

Theorem 4.2. *Assume that (CTP) is second-order KKT-invex. Then, every second-order stationary solution is an optimal solution.*

Proof. It is an immediate consequence of Theorem 4.1 and from the definition of second-order KKT-invexity. □

The converse of Theorem 4.2 is valid, as we see below.

Theorem 4.3. *Assume that every second-order stationary solution of (CTP) is an optimal solution and that there exists $\tilde{K} > 0$ such that*

$$\det(\nabla g^{I_a(t)}(\bar{z}(t),t)\nabla g^{I_a(t)}(\bar{z}(t),t)^\top) \geq \tilde{K} \ a.e \ in \ [0,T] \ \forall \bar{z} \in \Omega.$$

Then, (CTP) is second-order KKT-invex.

Proof. If (CTP) is not second-order KKT-invex, by Proposition 3.3, there exist $z, \bar{z} \in \Omega$ with $F(z) < F(\bar{z})$ such that, given $\gamma \in \mathscr{D}(\bar{z})$, (5)-(6) does not have any solution $(\eta, \omega) \in L^\infty([0,T];\mathbb{R}^n) \times L^\infty([0,T];\mathbb{R}_+)$. By Proposition 3.2, $\bar{z}$ is a second-order stationary solution. From the hypothesis, we see that $\bar{z}$ is an optimal solution, in contradiction to the existence of $z \in \Omega$ with $F(z) < F(\bar{z})$. □

Theorem 4.4. *Assume that there exists $\tilde{K} > 0$ such that*

$$\det(\nabla g^{I_a(t)}(\bar{z}(t),t)\nabla g^{I_a(t)}(\bar{z}(t),t)^\top) \geq \tilde{K} \ a.e \ in \ [0,T] \ \forall \bar{z} \in \Omega.$$

Then (CTP) is second-order KKT-invex if, and only if, every second-order stationary solution of (CTP) is an optimal solution.

Proof. It is an immediate consequence of Theorems 4.2 and 4.3. □

5 Final Considerations

The importance of second-order optimality conditions is well known in optimization theory: after obtaining the optimal solutions candidates through the application of the first-order necessary conditions, the second-order ones work as finer filter. Nevertheless, not all of the candidates which passed through the second filter are optimal solutions. In this work we presented a class of continuous-time optimization problems where this property occurs: the class of second-order KKT-invex problems. In addition, we showed, under a LICQ-type condition, that this is largest class of problems that possesses such a property. Relaxing the LICQ assumption is going to be topic of future work.

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References

[1] B. D. Craven and B. M. Glover. “Invex functions and duality”. In: **J. Austral. Math. Soc. Ser. A** 39 (1985), pp. 1–20.

[2] M. A. Hanson. “On sufficiency of the Kuhn-Tucker conditions”. In: **J. Math. Anal. Appl.** 80 (1981), pp. 545–550.

[3] V. I. Ivanov. “Second-order Kuhn-Tucker invex constrained problems”. In: **J. Global Optim.** 50.3 (2011), pp. 519–529.

[4] D. H. Martin. “The essence of invexity”. In: **J. Optim. Theory Appl.** 47.1 (1985), pp. 65–76.

[5] S. Nobakhtian and M. R. Pouryayevali. “KKT optimality conditions and nonsmooth continuous time optimization problems”. In: **Numer. Funct. Anal. Optim.** 32.11 (2011), pp. 1175–1189.

[6] V. A. de Oliveira. “Condições de Otimalidade e Dualidade em Otimização Não-Linear com Tempo-Contínuo”. Tese de Livre Docência. Universidade Estadual Paulista (Unesp). São José do Rio Preto. 2019.

[7] V. A. de Oliveira and M. A. Rojas-Medar. “Continuous-time multiobjective optimization problems via invexity”. In: **Abstr. Appl. Anal.** 2007.1 (2007). Art. ID 61296, 11 pp.

[8] V. A. de Oliveira and M. A. Rojas-Medar. “Continuous-time optimization problems involving invex functions”. In: **J. Math. Anal. Appl.** 327.2 (2007), pp. 1320–1334.

[9] M. A. Rojas-Medar, A. J. V. Brandão, and G. N. Silva. “Nonsmooth continuous-time optimization problems: sufficient conditions”. In: **J. Math. Anal. Appl.** 227.2 (1998), pp. 305–318.

