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Discrete Logistic Growth Model with Capability to Go Backward in Time, Based on Successive Operations

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Abstract. This paper aims to tackle the classic discrete logistic model for population growth using the formalism of successive mathematical operations (see [1]-[2]). This approach allows obtaining a closed-form expression with the capability of retro-action for generations before the first observed generation. Finally, to exemplify the advantages of this representation, it is used to compute the population size after and, outstandingly, before the reference, extending easily the usual discrete logistic growth model for all integer arguments.

keywords. Discrete Mathematics, Recursion, Successive Product, Successive Sum, Logistic Model.

1 Introduction

The logistic equation of population growth is one of the most known and used relation to model problems of ecology, and is applied in various areas of science. It was proposed by Verhulst [3] to solve the Malthusian unlimited exponential model, by introducing a regulation factor [4]. Although originally formulated in continuous time, there are several systems which are better characterized by discrete time considerations, when seasonal reproduction takes place or data are collected in spaced periods; so that the importance of its discrete version, treated here to circumvent several complexities normally found when searching solutions to this recursive-type equation [5]. From the mathematical point of view, the methodology of successive transformations introduced in [1], and successfully applied in [6]-[7], can contribute to deal with this kind of models even to run backwards in the generations, as done in the present work.

The classical discrete logistic growth equation assumes that the expected number of offspring varies with population size, such that

$$\mathrm{N}(t{+}1) \,=\, \mathrm{N}(t) + r\,\mathrm{N}(t)[\,1 - \mathrm{N}(t)/k\,];\ \forall\, t \in \mathbb{N} \text{ and } \forall\, k,\, r \in \mathbb{R}\,;\, k > 0 \text{ and } r \neq 0\,, \tag{1}$$

where $\mathrm{N}(t{+}1)$ is the number of individuals in the next generation, $\mathrm{N}(t)$ is the number of individuals in the current generation, r is the rate of growth, and k is the carrying capacity [4].

In this work, it will be proposed a new solution to (1) based on the successive operations defined in [1], including model generalization for any integer value of t. Then, the proposed solution is applied to go back and forward from the reference generation.

2 The Recursive Normalized Discrete Logistic Growth Model

Firstly, it will be recalled a key result on successive operations from [1]: If a mathematical model can be represented by the general form

$$\mathrm{G}(t+1) \,=\, \mathrm{F}(t+1)\,\mathrm{G}(t), \tag{2}$$

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with $\mathrm{F}, \mathrm{G} : \mathbb{Z} \to \mathbb{R}$, it is possible to search for a solution applying successive products, as defined in [1]. Then, the mathematical model (1) will be manipulated to obtain an equivalent form as in (2), in order to generate the proposed solution by the new successive operation methodology, with the extension from $\mathbb{N}$ to $\mathbb{Z}$. In this way, gathering the $\mathrm{N}(t)$ in equation (1) renders:

$$\mathrm{N}(t+1) \;=\; (1+r)\,\mathrm{N}(t) - (r/k)\,\mathrm{N}(t)^2;\; \forall\, t \in \mathbb{N} \text{ and } \forall\, k,\, r \in \mathbb{R}\,;\, k > 0 \text{ and } r \neq 0\,, \tag{3}$$

Multiplying (3) by $k/r\,;\; r \neq 0$ and reordering, leads to

$$\mathrm{N}(t)^2 \;=\; (k/r + k)\,\mathrm{N}(t) - (k/r)\,\mathrm{N}(t+1);\; \forall\, t \in \mathbb{N} \text{ and } \forall\, k,\, r \in \mathbb{R}\,;\, k > 0 \text{ and } r \neq 0\,. \tag{4}$$

To simplify (4) and obtain an expression similar to (2), it will be defined the coefficients

$$A\,(k,r) \;=\; k/r + k \quad \text{ and } \quad B\,(k,r) \;=\; -k/r\,;\; \forall\, k,\, r \in \mathbb{R}\,;\, k > 0 \text{ and } r \neq 0\,, \tag{5}$$

to obtain

$$\mathrm{N}(t)^2 \;=\; A\,\mathrm{N}(t) + B\,\mathrm{N}(t+1);\, \forall\, t \in \mathbb{N}\,,\, \forall\; A + B \neq 0\,,\; A/B \neq -1\,, \text{ and } B \in \mathbb{R}^*\,, \tag{6}$$

where $\mathbb{R}^* = \mathbb{R} - \{0\}$.

Expression (6) is a key in this methodology. It can be observed easily that with this expression one can go from t to $t+1$, as usual, however, it can be used to go back from $t+1$ to t, even for $t < 0$, by observing that it is a second-order polynomial equation in $\mathrm{N}(t)$. So, henceforth, it is possible to assume that expression (6) is valid for all $t \in \mathbb{Z}$. Owing this fact, after dividing (6) by $\mathrm{N}(t)$, a simple algebra allows to obtain:

$$\mathrm{N}(t+1) \;=\; \left[\frac{\mathrm{N}(t) - A}{B}\right]\mathrm{N}(t)\,;\, \forall\, t \in \mathbb{Z}\,,\; A + B \neq 0\,,\; A/B \neq -1\,, \text{ and } B \in \mathbb{R}^*\,. \tag{7}$$

By comparing (7) with (2), these expressions have the same form and we can identify

$$\mathrm{F}(t+1) \equiv (\,\mathrm{N}(t) - A\,)/B\,, \tag{8}$$

where $\mathrm{F}(t+1)$ is the growth factor. With this, it is possible to obtain the discrete logistic growth model solution by a successive product, with the advantage of an extension from $\mathbb{N}$ to $\mathbb{Z}$.

Firstly, the formalism of successive operations needs that the target sequence achieves the value 1 for $t = 0$. This fact forces the following normalization:

$$n(t) \;=\; \mathrm{N}(t)\,/\,\mathrm{N_R}\,;\; \mathrm{N_R} \;=\; \mathrm{N}(0) \text{ and } \mathrm{N_R} \in \mathbb{R}^*\,. \tag{9}$$

As a consequence of that normalization, in all the cases, one has $\quad n(0) \;=\; 1\,. \qquad (10)$

To solve (7) by applying successive operations, firstly it will be established the sequence indexed by integers of the normalized number of individuals in each generation:

$$(n(i))_{\,i\in\mathbb{Z}} \;=\; (\,\ldots\,,\, n(-2),\, n(-1),\, n(0) = 1,\, n(1),\, n(2),\, \ldots\,,\, n(i),\, \ldots\,);\, n(i) \in \mathbb{R}. \tag{11}$$

Multiplying and dividing (7) by $\mathrm{N_R}$, and making $n(t) = \mathrm{N}(t)/\mathrm{N_R}$ $\;\alpha = A/\mathrm{N_R}$, $\beta = B/\mathrm{N_R}$, and using (10), it is obtained the normalized discrete logistic growth model system

$$\begin{cases} n(0) = 1\,; \\ n(t+1) \;=\; \left[\dfrac{n(t) - \alpha}{\beta}\right] n(t)\,; \end{cases} \tag{12}$$

where $\alpha = A/\mathrm{N_R}$, $\beta = B/\mathrm{N_R}$ and considering $\forall\, t \in \mathbb{Z}\,,\; \alpha + \beta \neq 0\,,\; \alpha/\beta \neq -1\,$, and $\beta \in \mathbb{R}^*$.

Expression (12) can be seen as a recursion-equation system that satisfies a successive product as established in [1], particularized for $f : \mathbb{Z} \to \mathbb{R}$, and the usual multiplication, as follows:

Definition 2.1. *Let* $\mathrm{G}(\Phi,\cdot)$ *be the group where* Φ *is the set of all functions* $f:\mathbb{Z}\to\mathbb{R}$, SZ *the set of all sequences in the general form*

$$(f(i))_{i\in\mathbb{Z}} = (\,\dots,\, f(-2),\, f(-1),\, f(0),\, f(1),\, f(2),\, \dots,\, f(i),\, \dots);\ f(i)\in\mathbb{R}\,, \tag{13}$$

and '$\cdot$' *the usual arithmetic multiplication. Under these conditions, it is defined the function*

$$\text{Ᵽ}_{\mathbb{Z}}:\mathrm{SZ}\times\mathbb{Z}\to\mathrm{F}\ \ \textit{by}\qquad \begin{cases} f(i)\,!^{\,i\,=\,0} \;=\; 1\,;\ \text{ and} \\ f(i)\,!^{\,i\,=\,t+1} \;=\; f(t+1)\,f(i)\,!^{\,i\,=\,t}\,;\ \forall\, t\in\mathbb{Z} \end{cases} \tag{14}$$

where the notation used was introduced in [1]. By comparing (12) and (14), it is immediate to conclude that

$$(\,n(t)-\alpha\,)/\beta \;=\; f(t+1)\,, \tag{15}$$

and, after the assigning $t\leftarrow t-1$ in (15), it follows that

$$n(t) \;=\; \left[\frac{n(i-1)-\alpha}{\beta}\right]\Bigg!^{\,i\,=\,t}, \tag{16}$$

for $\forall\, t\in\mathbb{Z}\,,\ \alpha+\beta\neq 0\,,\ \alpha/\beta\neq -1\,$, and $\beta\neq 0\in\mathbb{R}\,$, which is the solution of (12).

3 Finding the Normalized Number of Individuals

The classical results for the usual positive values of t can be easily obtained by (16). Original results are now obtained here by the present methodology, that is the number of individuals for previous generations from the reference

If $t=0$ in (16), then $n(0)=1$. Now, making $t=-1$:

$$n(-1) \;=\; \left[\frac{n(i-1)-\alpha}{\beta}\right]\Bigg!^{\,i\,=\,-1} \;=\; \frac{\beta}{[\,n(-1)-\alpha\,]};\ \ \alpha+\beta\neq 0\,,\ \alpha/\beta\neq -1\,,\ \text{and}\ \beta\in\mathbb{R}^*\,. \tag{17}$$

By reordering (17):

$$n(-1)^2-\alpha\, n(-1)-\beta \;=\; 0\,;\ \alpha+\beta\neq 0\,,\ \ \alpha/\beta\neq -1\,,\ \ \beta\in\mathbb{R}^*\,. \tag{18}$$

Expression (18) is a quadratic equation in $n(-1)$ and to which the solution is

$$n(-1) \;=\; \frac{\alpha\mp\sqrt{\alpha^2+4\,\beta}}{2}\,;\ \forall\,\alpha\in\mathbb{R},\ \forall\,\beta\in\mathbb{R}^*\ \text{and}\ \alpha^2+4\,\beta\geqslant 0\,. \tag{19}$$

Making $t=-2$ in (16):

$$n(-2) \;=\; \left[\frac{n(i-1)-\alpha}{\beta}\right]\Bigg!^{\,i\,=\,-2} \;=\; \frac{\beta}{[\,n(-1)-\alpha\,]}\,\frac{\beta}{[\,n(-2)-\alpha\,]}\,. \tag{20}$$

In (20) it is possible to use (17), to get

$$n(-2) \;=\; \left[\frac{n(i-1)-\alpha}{\beta}\right]\Bigg!^{\,i\,=\,-2} \;=\; n(-1)\,\frac{\beta}{[\,n(-2)-\alpha\,]};\ \ \alpha+\beta\neq 0\,,\ \alpha/\beta\neq -1\,,\ \ \beta\in\mathbb{R}^*\,. \tag{21}$$

Reordering (21), one has

$$n(-2)^2-\alpha\, n(-2)-\beta\, n(-1) \;=\; 0\,;\ \alpha+\beta\neq 0\,,\ \ \alpha/\beta\neq -1\,,\ \text{and}\ \beta\in\mathbb{R}^*\,. \tag{22}$$

Expression (22) is a quadratic equation in $n(-2)$, then

$$n(-2) \;=\; \frac{\alpha\mp\sqrt{\alpha^2+4\,\beta\, n(-1)}}{2}\,;\ \forall\,\alpha\in\mathbb{R},\ \forall\,\beta\in\mathbb{R}^*\ \text{and}\ \alpha^2+4\,\beta\, n(-1)\geqslant 0\,. \tag{23}$$

By finite induction, it is possible to obtain the general equation

$$n(t)^2-\alpha\, n(t)-\beta\, n(t+1) \;=\; 0\,, \tag{24}$$

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whose solution allows to compute by recurrence the normalized number of individuals for all previous generations before the reference, that is

$$n(t) = \begin{cases} 1\,; & \text{if } \; t = 0\,; \text{ and} \\ \dfrac{\alpha \mp \sqrt{\alpha^2 + 4\,\beta\, n(t+1)}}{2}\,; & \text{if } \; t < 0\,; \end{cases} \tag{25}$$

where $\alpha\,,\,\beta \neq 0\,,\, \alpha + \beta \neq 0\,,\, \alpha/\beta \neq -1\,;$ and $\alpha^2 + 4\,\beta\, n(t+1) \geqslant 0\,.$

4 Application

The successive product closed formula will be applied to compute the number of individuals in the population after as well as before the reference generation $n(0)$.

Here, it will be considered an initial population composed of 200 individuals, a growth factor of 1.4 when there are 1 000 individuals, and 5 000 as the maximum acceptable number of individuals. This data have been taken from [8] (see also [9]), where it was analyzed using the standard way, that is, $t \geqslant 0$. Now, in the present application, the results of [8] will be extrapolated, since the successive product allows any integer value of t, and it will be also possible to answer questions about the previous generations $(t < 0)$ by the direct use of expression (16).

From (8), one has $\mathrm{F}(t{+}1) \; = \; (\,\mathrm{N}(t+1) - A\,)/B\,;$ and solving the system

$$\begin{cases} 1.4 \; = \; (\,1\,000 - A\,)/B\,, \\ 1.0 \; = \; (\,5\,000 - A\,)/B\,, \end{cases} \tag{26}$$

it is possible to find $A \, = \, 15\,000$ and $B \, = \, -10\,000$, which when normalized are $\alpha \, = \, 75$ and $\beta \, = \, -50$. By introducing these values in (16) one has

$$n(t) \; = \; \left[\frac{75 - n(i-1)}{50} \right] \Bigg|_{!}^{\,i=\,t} \; ; \; \forall\, t \in \mathbb{Z}\,, \tag{27}$$

as the successive product solution for the growth model of interest.

Table (1) reports the number of individuals, computed with (27) for $t \geqslant 0$ which are the same results obtained in [8].

Table 1: Values of $f(n(t-1))$, $n(t)$ and $\mathrm{N}(t)$, $0 \geqslant t \geqslant 17$, for $\alpha = 75$ and $\beta = -50$

t	$f(n(t-1))$	$n(t)$	$\mathrm{N}(t)$	t	$f(n(t-1))$	$n(t)$	$\mathrm{N}(t)$
0	unnecessary	1.000000	200.000	9	1.204763	17.784527	3556.90
1	1.480000	1.480000	296.000	10	1.144309	20.351003	4070.20
2	1.470400	2.176192	435.238	11	1.092980	22.243238	4448.65
3	1.456476	3.169572	633.914	12	1.055135	23.469624	4693.92
4	1.436608	4.553434	910.687	13	1.030607	24.187971	4837.59
5	1.408931	6.415476	1283.09	14	1.016240	24.580798	4916.16
6	1.371690	8.800047	1760.01	15	1.008384	24.786884	4957.38
7	1.323999	11.651254	2330.25	16	1.004262	24.892534	4978.51
8	1.266975	14.761846	2952.37	17	1.002149	24.946036	4989.21

To be more clear, Table(1) values are computed by the following steps:

Algorithm 1 Steps to compute the number os individuals, from (9), (14), and (15).

Input: α, β, N_{R}, $t \geqslant 0$,
Initialization: $i = 0$, $n(0) = 1.00000000$, $\mathrm{N}(0) = \mathrm{N}_{\mathrm{R}}$
while $(\, i < t\,)$ **do**
  $i \leftarrow i + 1$
  Normalized growth rate function: $$f(i) = (\, n(i-1) - \alpha\,)/\beta \tag{28}$$
  Normalized number of individuals: $$n(i) = f(i)\, n(i-1) \tag{29}$$
  Number of individuals: $$\mathrm{N}(i) = \mathrm{N}_{\mathrm{R}}\, n(i) \tag{30}$$
end while
Output: values of i, $f(i)$ **if** $t > 0$, $n(i)$, and $\mathrm{N}(i)$.

Now, to compute $n(t)$ for $t < 0$ one shall use (25) with $\alpha = 75$ and $\beta = -50$. There are two different possible values:

$$n_{\mp}(t) = \frac{75 \mp \sqrt{5625 - 200\, n(t+1)}}{2}\,; \qquad 5625 - 200\, n(t+1) \geqslant 0\,. \tag{31}$$

Table (2) shows the elements of two new possible sequences computed through (31), considering $-7 \leqslant t \leqslant 0$.

Table 2: Possible Values of $n_{\mp}(t)\,;\, -7 \leqslant t \leqslant 0$, for $\alpha = 75$ and $\beta = -50$:

t	$n_-(t)$	$n_+(t)$	t	$n_-(t)$	$n_+(t)$
0	1.000000000	74.00000000	-3	0.302003473	74.69799653
-1	0.672700343	74.32729966	-4	0.201879050	74.79812095
-2	0.451181087	74.54881891	-5	0.134828422	74.86517158

There are two reasons for not consider the $n_+(t)$ values. Firstly, as already commented before, the formalism of successive operations needs that $n(0) = 1$, which is not verified for $n_+(0)$. Secondly, the $n_+(t)$ values generate complex results insofar as t becomes more negative. Then, in Table (3) there are the reference and some previous generation numbers of individuals, obtained using only the $n_-(t)$ values.

Table 3: Values of $n(t)$ and $\mathrm{N}(t)$, $-17 \leqslant t \leqslant 0$, for $\alpha = 75$ and $\beta = -50$:

t	$n(t)$	$\mathrm{N}(t)$	t	$n(t)$	$\mathrm{N}(t)$
0	1.000000000	200.0000000	-9	0.026709904	5.3419808
-1	0.672700343	134.5400686	-10	0.017810832	3.5621665
-2	0.451181087	90.2362174	-11	0.011875769	2.3751537
-3	0.302003473	60.4006946	-12	0.007918015	1.5836030
-4	0.201879050	40.3758100	-13	0.005279048	1.0558096
-5	0.134828422	26.9656844	-14	0.003519531	0.7039061
-6	0.089993595	17.9987190	-15	0.002346427	0.4692844
-7	0.060043800	12.0087601	-16	0.001564317	0.3128635
-8	0.040050588	8.0101175	-17	0.001042893	0.2085785

In Figure (1) it is shown all the results from Table (1) and Table (3) , covering all generations, that is, the reference, the subsequent and the previous ones.

To be more clear, Table (3) values are computed by the following steps:

Algorithm 2 Steps to compute the number of individuals, from (9), and (25).

Input: α, β, N_{R}, $t \leqslant 0$,

Initialization: $i = 0$, $n(0) = 1.00000000$, $\mathrm{N}(0) = \mathrm{N}_{\mathrm{R}}$

while $(\, i > t\,)$ **do**

  $i \leftarrow i - 1$

  if $\left(\alpha^2 + 4\,\beta\, n(i+1) \geqslant 0\right)$ **then**

    1st root:
$$n_-(t) \;=\; \frac{\alpha - \sqrt{\alpha^2 + 4\,\beta\, n(t+1)}}{2} \tag{32}$$

    2nd root:
$$n_+(t) \;=\; \frac{\alpha + \sqrt{\alpha^2 + 4\,\beta\, n(t+1)}}{2} \tag{33}$$

    Choose $n(i)$: $n(i) = n_-(t)$, **or** $n(i) = n_+(t)$

    Number of individuals:
$$\mathrm{N}(i) \;=\; \mathrm{N}_{\mathrm{R}}\, n(i) \tag{34}$$

  end if

end while

Output: values of i, $n(i)$, and $\mathrm{N}(i)$.

It should be noted that it was used the same original setup of [8]. Despite this fact, the mathematical model solution proposed here provides a large range of new results. For example, by direct inspection of Table (3) it will be possible to conclude that the first individual arose in the thirteenth generation prior to the reference, as highlighted in Figure (2).

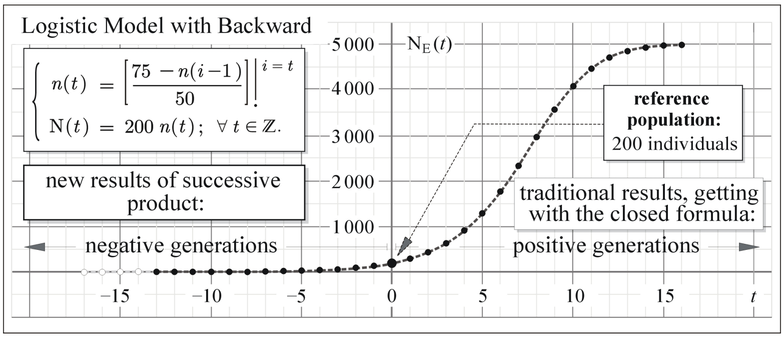

Figure 1: All the results for $\mathrm{N}(t)$: the usual set and the extension.

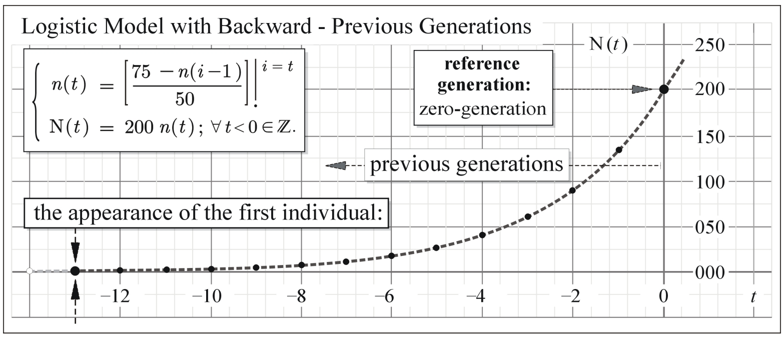

Figure 2: Detail of the extended results for $\mathrm{N}(t)$, highlighting the most probable generation where the first individual arose.

5 Final Considerations

In this work, the discrete logistic equation has been solved by a new approach considering successive operations, based on [1], in particular by the successive product methodology, which made possible to step backwards through the generations for building automatically the extensions for past generations relative to the reference one. The expressions obtained with this method are simple and compact. The results obtained are consistent with the expected values since from any previous generation it is possible to determine the subsequent amount of individuals. This method generated an extension to the integer set, capable of directly running backward in the generations to capture the most probable generation in which the population started to grow. Besides, this work constitutes a basis for modeling other kinds of discrete logistic growth, by generalizing (16), with the additional advantage of providing an extension from the natural domain to the integer one.

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