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First-order methods for the convex-hull membership problem and applications

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Let $\mathcal{A} \coloneqq \{v_1, v_2, \ldots, v_n\}$ be a finite subset of $\mathbb{R}^m$ and consider a given point $p \in \mathbb{R}^m$. The **convex hull membership problem** (CHMP) consists in deciding whether $p \in \text{conv}(\mathcal{A})$, where $\text{conv}(\mathcal{A})$ denotes the convex hull of $\mathcal{A}$. This problem is related to fundamental concepts in linear programming and finds important applications in computational geometry [3, 4].

One can formulate CHMP as

$$\min_{x \in \Delta_n} \quad \frac{1}{2}\|Ax - p\|^2, \tag{1}$$

where $\Delta_n \coloneqq \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1, x \geq 0\}$ is the unit simplex in $\mathbb{R}^n$ and A is the matrix whose columns are the vectors in $\mathcal{A}$. Another possible formulation is given by

$$\begin{aligned} \min_{y \in \mathbb{R}^m} \quad & \frac{1}{2}\|y - p\|^2 \\ \text{s.t.} \quad & y \in \text{conv}(\mathcal{A}). \end{aligned} \tag{2}$$

Notice that the optimal value for (1) and (2) is zero if and only if $p \in \text{conv}(\mathcal{A})$.

These mathematical programming formulations suggest the use of first-order methods such as projected gradient and conditional gradient to solved CHMP.

More recently, a geometric algorithm has been proposed based on the following theorem of alternatives [4, Theorem 4].

Theorem 1 *For a given set $\mathcal{A} \coloneqq \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^m$ and a point $p \in \mathbb{R}^m$, precisely one of the two conditions is satisfied:*

1. *For all $p' \in \text{conv}(\mathcal{A})$, there exists $v_i \in \mathcal{A}$ such that $\|p' - v_i\| \geq \|p - v_i\|$;*

2. *There exists $p' \in \text{conv}(\mathcal{A})$ such that $\|p' - v_i\| < \|p - v_i\|$, for all $i = 1, \ldots, n$.*

This result induces the following algorithm. Given $p_k \in \text{conv}(\mathcal{A})$, choose a $v_i \in \mathcal{A} \setminus \{p_k\}$ such that $\|v_i - p\| \leq \|v_i - p_k\|$ and define p_{k+1} as the point in the line segment $p_k v_i$ closest to p. If such v_i does not exist, then it is possible to show that the hyperplane given by the equation $(p - p_k)^T y = (\|p\|^2 - \|p_k\|^2)/2$ separates p from $\text{conv}(\mathcal{A})$, and thus $p \notin \text{conv}(\mathcal{A})$. Since each iteration involves the triangle with vertices p, p_k and v_i, this algorithm is called Triangle Algorithm (TA).

In this work, we first discuss similarities and differences between TA and conditional gradient (also know as Frank-Wolfe [6]). Then, on the basis of Theorem 1, we develop suitable stopping

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criteria for CHMP to be integrated into other first-order methods, allowing a fair numerical comparison between those and TA. Our numerical study reveals that there is no clear winner, and the method of choice depends on the geometry of $\text{conv}(\mathcal{A})$ as well as on the location of p.

Finally, we consider as an application the so-called irredundancy problem, which consists in determining all the extreme points of $\text{conv}(\mathcal{A})$. This is important, for example, in determining Minimum Volume Enclosing Ellipsoid (MVEE) [5] and Convex Hull Approximation [2], and in other problems in data science.

To address the irredundancy problem, we consider a scheme proposed in [1] that can be described as follows. Choose a $v \in \mathcal{A}$ and return a point in $\mathcal{A}$ that is farthest from v, say v' (this v' is an extreme point of $\mathcal{A}$ [1, Proposition 3]). Add v' to the working set $\hat{\mathcal{A}}$. Now, randomly select a point $v \in \mathcal{A} \backslash \hat{\mathcal{A}}$ and determine if $v \in \text{conv}(\hat{\mathcal{A}})$. If $v \in \text{conv}(\hat{\mathcal{A}})$ we remove v from $\mathcal{A}$. Otherwise, we can build a hyperplane, say, described by $c^T y = \eta$, which separates v from $\text{conv}(\hat{\mathcal{A}})$. Define the set $\mathcal{A}' \coloneqq \text{argmax}\{c^T y,\ y \in \mathcal{A} \setminus \hat{\mathcal{A}}\}$. Choose $u \in \mathcal{A}'$ and obtain $w \in \mathcal{A}'$ farthest from u. Add w to $\hat{\mathcal{A}}$ and remove it from $\mathcal{A}$. Repeat this process until $\mathcal{A}$ is empty. $\hat{\mathcal{A}}$ will be an approximate set of extreme points of $\text{conv}(\mathcal{A})$. Notice that in each step of this scheme, a CHMP needs to be solved, and we could apply the first-order methods mentioned above.

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