Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

Estabilidade Assintótica em Modelos de Sistemas Descontínuos

Durval José Tonon
Instituto de Matemática e Estatística, UFG
74001-970, Goiânia, GO
E-mail: djtonon@ufg.br,

Tiago de Carvalho,
Dpto de Mat., Fac. de Ciências, UNESP,
17033-360, Bauru, SP
E-mail: tcarvalho@fc.unesp.br,

Marco A. Teixeira,
IMECC, UNICAMP
13081–970, Campinas, SP
E-mail: teixeira@ime.unicamp.br.

Abstract: *This work deals with nonsmooth three-dimensional vector fields exhibiting a typical singularity at the origin. We focus on a class of generic one parameter families* Z_λ *of Filippov systems and address the persistence problem for the asymptotic stability at the singularity near the bifurcation point* ($\lambda = 0$).

Keyword: *piecewise smooth vector fields, cusp-fold singularity, asymptotic stability.*

1 Introduction

We analyze the bifurcation diagram and the asymptotic stability of the following family of piecewise smooth vector fields (PSVFs for short):

$$Z_\lambda(x,y,z) = \begin{cases} X^\lambda_{a,b}(x,y,z) = (a, \lambda, b(y+x^2)) & \text{if } z \geq 0, \\ Y_{c,d}(x,y,z) = (c,d,x) & \text{if } z \leq 0, \end{cases} \tag{1}$$

or, equivalently, $Z_\lambda(x,y,z) = (\dot{x},\dot{y},\dot{z}) = \frac{1}{2}\Big((a+c, \lambda+d, b(y+x^2)+x) + sgn(z)\,(a-c, \lambda-d, b(y+x^2)-x)\Big)$, with $a,b,c,d,\lambda \in \mathbb{R}, b\cdot c \neq 0$ and λ arbitrarily small. The main result is:

Theorem A. Let Z_λ given by (1). If $a<0$, $b<0$, $c>0$, $d>0$ and $a+bd>0$ then:

- Z_λ is asymptotically stable at the origin when $\lambda \geq 0$ and
- Z_λ is not Lyapunov stable at the origin when $\lambda < 0$.

This work is organized as follows: In Section 2 we give an overall description of the problem. In Section 3 we formalize some basic concepts on PSVFs. In Section 4 some auxiliary results are stated and we pave the way in order to prove the main results in Section 5. In Section 6 we illustrated our main result considering a small perturbation of a relay system.

2 Setting the problem

In our approach two smooth vector fields on both sides of the **switching region** Σ are chosen to be arbitrary vector fields with no relation between them. The concatenation of its trajectories give rise to nonsmooth dynamical systems which are widely used to model various dynamical behaviors in Electrical and Electronic Engineering, Physics, Economics, among other areas. So, such systems can provide dynamical models at a level of abstraction that is appropriate for several purposes (for a wide range of applications, see for instance [2] and references therein). It is worth mentioning that in [1] the asymptotic stability of a class of relay systems in $\mathbb{R}^n$ is discussed. They have the form:

$$\dot{x} = Ax + sgn(x_1)k,$$

where $x = (x_1, x_2, \ldots, x_n)$, A is an $n \times n$ real valued matrix and $k = (k_1, k_2, \ldots, k_n)$ is a constant vector in $\mathbb{R}^n$.

In our approach we use tools involving the non transversal contact between a vector field and the boundary Σ of a manifold (see [6]). This technique permits to deal the rather complicated local dynamics near the singularity into a simpler local dynamics (see [3] for a planar analysis on this subject). In the 3-dimensional case, there are two important distinguished generic singularities: the points where this contact is either quadratic or cubic, which are called fold and cusp singularities, respectively.

As it is fairly known, from a generic cusp singularity emanate two branches of fold singularities, see Figure 1, one of such branches formed by **visible fold singularities**, where the trajectories tangent are visible and one of such branches composed by **invisible fold singularities**, where the trajectories tangent are not visible. Moreover, it is possible for a point $p \in \Sigma$ be a tangency point for both X and Y. When p is a fold singularity of both X and Y we say that p is a **two-fold singularity** and when p is a cusp singularity for X and a fold singularity for Y we say that p is a **cusp-fold singularity**, see Figure 1 below. In [5] two-fold singularities

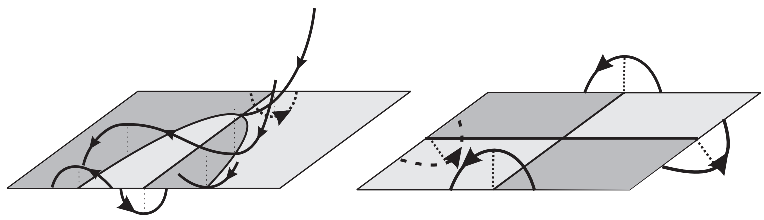


Figura 1: On the left it appears a cusp-fold singularity and on the right a two-fold singularity.

are studied and in [2] applications of such theory in electrical and control systems, respectively, are exhibited. We stress that this singularity is particularly relevant because in its neighborhood some of the key features of a piecewise smooth system are present.

3 Basic Theory about PSVFs

3.1 Filippov's Convention

Let $K = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 < \delta\}$, where $\delta > 0$ is arbitrarily small. Consider $\Sigma = \{(x, y, z) \in K \,|\, z = 0\}$. Clearly the switching manifold Σ is the separating boundary of the regions $\Sigma_+ = \{(x, y, z) \in K \,|\, z \geq 0\}$ and $\Sigma_- = \{(x, y, z) \in K \,|\, z \leq 0\}$.

Designate by $\mathfrak{X}^r$ the space of C^r-vector fields on K endowed with the C^r-topology with $r = \infty$ or $r \geq 1$ large enough for our purposes. Call Ω^r the space of vector fields $Z : K \to \mathbb{R}^3$ such that

$$Z(x, y, z) = \begin{cases} X(x, y, z), & \text{for} \quad (x, y, z) \in \Sigma_+, \\ Y(x, y, z), & \text{for} \quad (x, y, z) \in \Sigma_-, \end{cases}$$

where $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ are in $\mathfrak{X}^r$. We may consider $\Omega^r = \mathfrak{X}^r \times \mathfrak{X}^r$ endowed with the product topology and denote any element in Ω^r by $Z = (X, Y)$, which we will accept to be multivalued in points of Σ. The basic results of differential equations, in this context, were stated by Filippov in [4]. Related theories can be found in [6] and references therein. On Σ we distinguish the following regions:

- **Crossing Region:** $\Sigma^c = \{p \in \Sigma \,|\, X_3(p) \cdot Y_3(p) > 0\}$. Moreover, we denote $\Sigma^{c+} = \{p \in \Sigma \,|\, X_3(p) > 0, Y_3(p) > 0\}$ and $\Sigma^{c-} = \{p \in \Sigma \,|\, X_3(p) < 0, Y_3(p) < 0\}$.
- **Sliding Region:** $\Sigma^s = \{p \in \Sigma \,|\, X_3(p) < 0, Y_3(p) > 0\}$.
- **Escaping Region:** $\Sigma^e = \{p \in \Sigma \,|\, X_3(p) > 0, Y_3(p) < 0\}$.

When $q \in \Sigma^s$, following the Filippov's convention, the **normalized sliding vector field** associated to $Z \in \Omega^r$ is the vector field Z^s tangent to Σ^s expressed in coordinates as

$$Z^s(q) = ((X_1Y_3 - Y_1X_3)(q), (X_2Y_3 - Y_2X_3)(q)). \tag{2}$$

We define the sets of tangential singularities $S_X = \{p \in \Sigma | X_3(p) = 0\}$ and $S_Y = \{p \in \Sigma | Y_3(p) = 0\}$.

3.2 The first return map

Consider $p \in \Sigma$ and suppose that there exists $t_1(p)$, the positive return time of trajectory of X passing through p. We put $\phi_X(t_1(p), p) = p_1 \in \Sigma$. Let $t_2(p_1)$ be the positive return time of trajectory of Y passing through $p_1 \in \Sigma$. The *first return map* associated to $Z = (X, Y)$ is defined by the composition $\varphi_Z(p) = \phi_Y(t_2(p_1), \phi_X(t_1(p), p))$. Considering the PSVF given in (1), we get the expression of the first return map

$$\varphi_{Z_\lambda}(x, y) = \left(\frac{2ax + \Delta_1}{4a}, y + \frac{d(2ax + \Delta_1)}{2ac} + \frac{\lambda(-6ax - \Delta_1)}{4a^2}\right) \tag{3}$$

where $\Delta_1 = 3\lambda - \sqrt{9\lambda^2 + 36a\lambda x - 12a^2(x^2 + 4y)}$.

4 Auxiliary Results

4.1 The case $\lambda = 0$

Consider (1), with $\lambda = 0$. In this case, Z_0 present a cusp-fold singularity at the origin. Note that $S_X = \{(x, y, 0) \in \Sigma \,|\, y = -x^2\}$ and $S_Y = \{(x, y, 0) \in \Sigma \,|\, x = 0\}$ are the sets of tangential singularities of X and Y respectively.

4.1.1 Local dynamics of the normalized sliding vector field

By (2), the normalized sliding vector field is given by $Z_0^s = (ax - bc(y + x^2), -db(y + x^2))$. So, the eigenvalues of Z_0^s are $\lambda_1^0 = a$ and $\lambda_2^0 = -db$ and the eigenspaces associated to λ_1^0 and λ_2^0, respectively, are $E_1^0 = \{(x, y, 0) \in \Sigma \,|\, y = 0\}$ and $E_2^0 = \left\{(x, y, 0) \in \Sigma \,|\, y = \frac{(a+bd)x}{bc}\right\}$. In order to get Z_0 asymptotically stable at the cusp-fold singularity, some hypotheses must be imposed in the parameters:

Hypothesis 1 (H_1): The fold point generated by the vector field Y must be invisible. So, $c > 0$.
Hypothesis 2 (H_2): The origin must be asymptotically stable for Z_0^s. So $\lambda_1^0 = a < 0$ and $-\lambda_2^0 = bd > 0$.

Following H_1 and H_2, the phase portraits of Z_0, in Σ^s, is given by one of the following illustrations, in Figure 2. However, just at Case (a) of Figure 2 we hope asymptotic stability. In fact, in Case (b), it is easy to see that the smooth vector fields X^0 is not Lyapunov stable at the origin. So we consider the next hypothesis:

Hypothesis 3 (H_3): The cusp singularity generated by the vector field X^0 must be of the topological type described in Figure 2, Case (a). So, $b < 0$. By consequence of H_2 and H_3 we

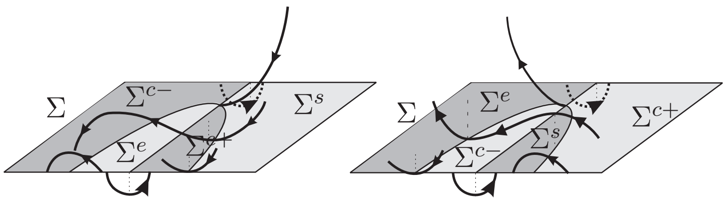


Figura 2: The two possible local dynamics of Z_0 with hypothesis H_1 and H_2.

conclude that $d < 0$. Faced to H_2, in order to obtain that the sliding region Σ^s is invariant for Z_0^s and the origin is asymptotically stable, we impose the following condition:
Hypothesis 4 (H_4): E_2^0 is stronger than E_1^0, i.e., $|\lambda_1| < |\lambda_2|$. So, $-a < bd \Rightarrow 0 < a + bd$. As an immediate consequence of H_4, we get $(bc)/(a + bd) < 0$ and $E_2^0 \cap \Sigma^s = \emptyset$, see Figure 3.

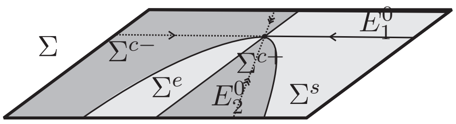


Figura 3: Local dynamic of Z_0^s.

4.1.2 Local dynamics of the first return map

Now, in order to determine the dynamics of the positive trajectories of Z_0 we consider the first return map, given in (3), with $\lambda = 0$. We get

$$\varphi_{Z_0}(x, y) = \left(\frac{ax - \sqrt{-3a^2(x^2 + 4y)}}{2a}, y + \frac{d(ax - \sqrt{-3a^2(x^2 + 4y)})}{ac} \right).$$

Given a point $p \in \mathbb{R}^3$, it is easy to see that the positive trajectory $\phi^+_{Z_0}(p)$ of Z passing through p intersects $\overline{\Sigma^s} \cup \overline{\Sigma^{c+}}$. In what follows we prove that $\phi^+_{Z_0}(p) \cap \overline{\Sigma^s} \neq \emptyset$.

Lemma 1.

$$\varphi_{Z_0}(\Sigma^{c+}) \subset \left\{ (x, y, 0) \in \Sigma \,|\, -\frac{x^2}{3} + 2\frac{d}{c}x < y < -\frac{x^2}{4} + 2\frac{d}{c}x, \text{ with } x > 0 \right\}.$$

Demonstração. Given a point $p_0 = (x_0, y_0, 0) \in \Sigma^{c+}$ (where $x_0 > 0$ and $y_0 < 0$), the trajectory of X_0 by p_0 intersects Σ at $p_1 \in \Sigma^{c-}$ and the trajectory of Y by p_1 intersects Σ at p_2, where p_2 is situated between the curves $y = -\frac{x^2}{3} + 2\frac{d}{c}x$ and $y = -\frac{x^2}{4} + 2\frac{d}{c}x$ which correspond to the images of the curves $x = 0$, with $y < 0$ and $y = -x^2$, with $x > 0$, respectively. □

Lemma 2. *Given $p_0 = (x_0, y_0, 0) \in \overline{\Sigma^{c+}}$, call $p_1 = (x_1, y_1, 0) = \varphi_{Z_0}(p_0)$ and $p_n = (x_n, y_n, 0) = \varphi^n_{Z_0}(p_0)$, when it is well defined. Then $x_1 > x_0$ and $x_n \to \infty$ when $n \to \infty$.*

Demonstração. Given $p_0 = (x_0, y_0, 0) \in \overline{\Sigma^{c+}}$, a straightforward calculus show that $x_1 = \frac{x_0}{2} + \frac{\sqrt{3}\sqrt{-(x_0^2+4y_0)}}{2}$ where $p_1 = (x_1, y_1, 0) = \varphi_{Z_0}(p_0)$. Since $p_0 \in \overline{\Sigma^{c+}}$ we conclude that $y_0 \leq -x_0^2 < -x_0^2/3$. So,

$$y_0 < -x_0^2/3 \Rightarrow -4x_0^2 - 12y_0 > 0 \Rightarrow (-3(x_0^2 + 4y_0)) > x_0^2 \Rightarrow$$

$$\frac{\sqrt{-3(x_0^2 + 4y_0)}}{2} > \frac{x_0}{2} \Rightarrow x_1 > x_0.$$

A recursive analysis proves that $x_{n+1} > x_n$. In fact, repeating the previous argument $x_{n+1} = \frac{x_n + \sqrt{-3(x_n^2+4y_n)}}{2} > 2x_n \Rightarrow \frac{x_{n+1}}{x_n} > 2$. Since, $\frac{x_{n+1}}{x_n} > 1$, by a test of convergence of sequences, we get $x_n \to \infty$. □

DOI: 10.5540/03.2015.003.01.0035

Proposition 1. *For all $p \in K$ it happens $\phi^+_{Z_0}(p) \cap \overline{\Sigma^s} \neq \emptyset$.*

Demonstração. As we observed above, given a point $p \in K$, it is easy to see that $\phi^+_{Z_0}(p) \cap [\overline{\Sigma^s} \cup \overline{\Sigma^{c+}}] \neq \emptyset$. So, it is enough to prove that $\varphi^{n_0}_{Z_0}(\overline{\Sigma^{c+}}) \subset \overline{\Sigma^s}$ for some $n_0 > 0$. By Lemma 1 we obtain that $\varphi_{Z_0}(\overline{\Sigma^{c+}}) \subset \left\{(x,y,0) \in \Sigma \,|\, \frac{x^2}{3} + 2\frac{d}{c}x \leq y \leq -\frac{x^2}{4} + 2\frac{d}{c}x, \text{ with } x > 0\right\}$. By Lemma 2, there exists $n_0 > 0$ such that $p_{n_0} = (x_{n_0}, y_{n_0}, 0) = \varphi^{n_0}_{Z_0}(p)$ satisfies $y_{n_0} + x^2_{n_0} \geq 0$, since $y_{n_0-1} > \frac{-64d^2}{9c^2}$ by Lemma 1. Therefore $p_{n_0} \in \overline{\Sigma^s}$. □

4.2 The case $\lambda \neq 0$

When $\lambda \neq 0$, we consider the normal form (1), presenting a fold-fold singularity at the origin, since $b \neq 0$. The local dynamics for Z_λ is given in Figure 4.

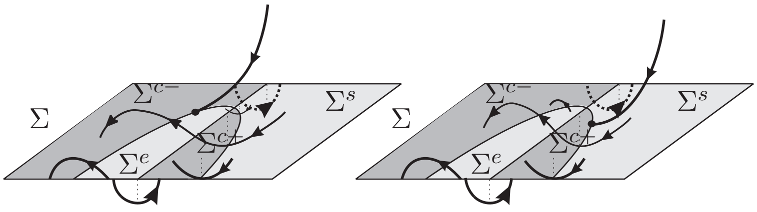


Figura 4: The local dynamic of Z_λ, with hypotheses H_1 and H_3.

4.2.1 Local dynamics of the normalized sliding vector fields

According to (2), the normalized sliding vector field is given by $Z^s_\lambda = (ax - bc(y + x^2), \lambda x - db(y + x^2))$. Let $\Delta_3 = (a + bd)^2 - 4bc\lambda$. The eigenvalues of $DZ^s_\lambda(0,0)$ are $\lambda^\lambda_1 = \frac{a-bd-\sqrt{\Delta_3}}{2}$ and $\lambda^\lambda_2 = \frac{a-bd+\sqrt{\Delta_3}}{2}$ and the eigenspaces associated to λ^λ_1 and λ^λ_2, respectively, are $E^\lambda_1 = \left\{(x,y,0) \in \Sigma \,|\, y = \frac{2\lambda}{a+bd-\sqrt{\Delta_3}}x\right\}$ and $E^\lambda_2 = \left\{(x,y,0) \in \Sigma \,|\, y = \frac{2\lambda}{a+bd+\sqrt{\Delta_3}}x\right\}$. Under the hypotheses $H_1 - H_4$, we get that $\lambda^\lambda_{1,2}$ are negative and E^λ_1 is stronger than E^λ_2. Besides, we obtain the following results:

Lemma 3. *The eigenspace $E^\lambda_1 \subset \Sigma^c, E^\lambda_2 \subset [\Sigma^s \cup \Sigma^e]$, when $\lambda > 0$, $E^\lambda_2 \subset \Sigma^c$, when $\lambda < 0$, see Figure 5.*

Demonstração. Straightforward according to expressions of $E^\lambda_{1,2}$. □

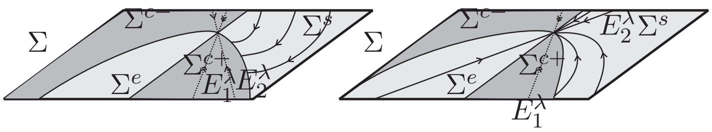


Figura 5: The local dynamic of Z^s_λ with hypothesis $H_1 - H_4$.

Note that in case $\lambda < 0$, the sliding vector fields has a transient behavior in Σ^s, as consequence all the obits in Σ^s will be iterated by the first return map, whereas in case $\lambda > 0$, Z^s_λ is asymptotic stable at the origin.

4.2.2 Local dynamics of the first return map

Now, in order to determine the dynamics of the positive trajectories of Z_λ, we consider the first return map φ_{Z_λ} of Z_λ, whose expression is given in (3).

Lemma 4. *Under the hypothesis $H_1 - H_4$ the origin is a hyperbolic saddle fixed point for $\varphi_{Z_\lambda}, S^\lambda_\pm \subset \Sigma^c$ when $\lambda > 0$ and $S^\lambda_+ \subset \Sigma^c, S^\lambda_- \subset [\Sigma^e \cup \Sigma^s]$ when $\lambda < 0$. Besides, S^λ_+ (resp. S^λ_-) is an expansive (resp. contractive) direction.*

Demonstração. Follows by the expressions of the eigenvalues and the eigenspaces of $D\varphi_{Z_\lambda}(0)$, respectively. □

By Lemma 4, when $\lambda > 0$, we get that given $p \in \Sigma^{c+}$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}_{Z_\lambda}(p) \in \Sigma^s$. And Lemma 3, under the hypothesis $H_1 - H_4$, provides that Z^s_λ is asymptotically stable at the origin. See Figure 6, when the dotted lines in Σ^{c+} represent the iterated of φ_{Z_λ} and the line in Σ^s the dynamic of Z^s_λ. The dynamics of Z_λ is given in Figure 6. In this case,

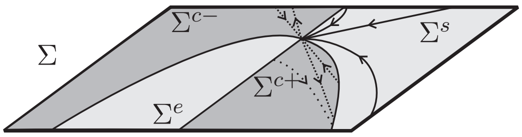


Figura 6: Dynamic of φ_{Z_λ} and Z^s_λ, under the hypothesis $H_1 - H_4$ with $\lambda > 0$.

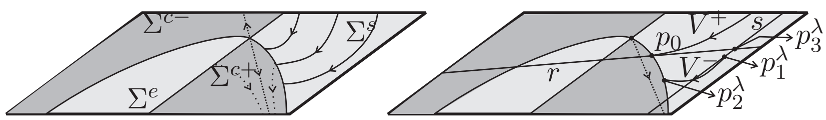
we get that Z_λ is asymptotic stable at the origin, under the hypothesis $H_1 - H_4$. When $\lambda < 0$, Lemma 3 provides that the trajectories of the sliding vector field Z^s_λ have a transient behavior in Σ^s.

We denote $\varphi_{Z_\lambda}(p_0) = p^\lambda_1 = (x^\lambda_1, y^\lambda_1, 0)$, that can be situated at $\overline{\Sigma^{c+}}$ and in this case, by Lemma 4, its distance to the origin increase when compared with p_0. Otherwise, p^λ_1 can be situated at Σ^s and in this case the trajectory by this point slides to the parabola $y = -x^2$. The intersection point will be called $p^\lambda_2 = (x^\lambda_2, y^\lambda_2, 0) = (x^\lambda_2, -(x^\lambda_2)^2, 0)$. As the origin is an attractor for Z^s_λ we have to discuss the behavior of the mapping φ_{Z_λ} at the origin and answer if the attractiveness of it is greater or less than the repulsiveness of the first return map φ_{Z_λ}.

Denote by $d(p, 0)$ the euclidian distance between the point p to the origin.

Lemma 5. *Under the hypotheses $H_1 - H_4$ with $\lambda < 0$ and with the previous notation, $d(p^\lambda_2, 0) > d(p_0, 0)$.*

Demonstração. In Subsection 4.2.1 we get the expression of Z^S_λ. The straight line $r : (x(\alpha), y(\alpha), 0) = (x_0, -x_0^2, 0) + \alpha(ax_0, \lambda x_0, 0)$ with $\alpha \in \mathbb{R}$, is tangent to the trajectory of Z^s_λ by $p_0 = (x_0, -x_0^2, 0)$. Note that r splits Σ^s in two regions, denoted by V^+ and V^-. Consider the vertical straight line $s : p = p^\lambda_1 + \beta(0, 1, 0)$, with $\beta \in \mathbb{R}$, see Figure 7. We get that $r \cap s = p^\lambda_3$, where $p^\lambda_3 = (x^\lambda_3, y^\lambda_3, 0) = \left(x^\lambda_1, -x_0^2 + \frac{\lambda}{a}\left(\frac{3\lambda}{2a} + x_0\right), 0\right)$. Comparing y^λ_3 with y^λ_1 we get $y^\lambda_1 < y^\lambda_3$. Therefore p^λ_1, and consequently p^λ_2, are situated at the region V^- described in Figure 7. So, $d(p^\lambda_2, 0) > d(p_0, 0)$. □



(a) (b)

Figura 7: In (a) we have the local dynamic of φ_{Z_λ} (dotted line) and Z^s_λ (in Σ^s). In (b) is presented the straight lines r and s, the points $p_0, p^\lambda_1, p^\lambda_2$ and p^λ_3 and the regions V^+ and V^-.

DOI: 10.5540/03.2015.003.01.0035

5 Proof of main result

When $\lambda = 0$, by Proposition 1, the trajectories of all points in $\mathbb{R}^3$ intersect $\overline{\Sigma^s}$. By hypotheses H_2 and H_4, the ω-limit set of all trajectories in $\overline{\Sigma^s}$ is the origin. So, Z_0 is asymptotically stable at the origin.

When $\lambda > 0$, we get that the trajectories of all points in K intersect $\overline{\Sigma^s}$. Moreover, the origin is a hyperbolic attractor for Z_λ^s and by Lemma 3 we get $E_1^\lambda \subset \Sigma^c$ and $E_2^\lambda \subset \Sigma^s$, for $x > 0$. Therefore, the positive orbits of Z_λ follows the orbits of Z_λ^s. So, Z_λ is asymptotically stable at the origin. In case when $\lambda < 0$, the result is an immediate consequence of Lemmas 3, 4 and 5.

6 Asymptotic stability in a perturbed relay system

As an illustration of our main result, let us consider a model that is a small perturbation of the relay system expressed as $z''' = \alpha\, sgn(z)$ where $\alpha \in \mathbb{R}$ (see [1]). This system can be written as

$$Z_0(x,y,z) = \begin{cases} X(x,y,z) = (y,\alpha,x) & \text{if } z \geq 0, \\ Y(x,y,z) = (y,-\alpha,x) & \text{if } z \leq 0. \end{cases}$$

Now we consider the following vector field

$$Z_\varepsilon(x,y,z) = \begin{cases} X(x,y,z) = (y,\alpha,x) & \text{if } z \geq 0, \\ Y_\varepsilon(x,y,z) = (y,-\alpha,x+\varepsilon y) & \text{if } z \leq 0, \end{cases} \tag{4}$$

where ε is arbitrarily small. This system presents a fold-cusp singularity at the origin. Applying the change of coordinates $(u,v,w) = \left(\frac{a}{\alpha}y, \frac{2a^2}{\alpha}x - \frac{a^2}{\alpha^2}y^2, \frac{2a^2b}{\alpha}z\right)$ in (4) we obtain the PSVF

$$Z_\varepsilon(u,v,w) = \begin{cases} X(u,v,w) = (a,0,b(v+u^2)) & \text{if } \dfrac{\alpha}{b}w \geq 0, \\ \\ Y_\varepsilon(u,v,w) = (-a,4au,2ab\varepsilon u + bv + bu^2) & \text{if } \dfrac{\alpha}{b}w \leq 0. \end{cases} \tag{5}$$

The system (5) is Σ-equivalent to the system (1) with $\lambda = 0$. As consequence there exists a convenient choice of parameters such that the perturbed relay system (4) is asymptotic stable at the origin.

Acknowledgments. The first author is partially supported by a FAPESP-BRAZIL grant 2012/00481-6. This work is partially realized at UFG as a part of project numbers 35796 and 35797. The third author is supported by FAPEG-BRAZIL grant 2012/10267000803 and a CNPq-BRAZIL grant 478230/2013-3.

Referências

[1] D.V. Anosov, *Stability of the equilibrium positions in relay systems, Automation and remote control*, vol. XX, 2, (1959).

[2] M. di Bernardo, A. Colombo and E. Fossas, *Two-fold singularity in nonsmooth electrical systems*, Proc. IEEE International Symposium on Circuits ans Systems, (2011), 2713–2716.

[3] C.A. Buzzi, T. de Carvalho and M.A. Teixeira, *Birth of limit cycles from a nonsmooth center*, Journal de Mathématiques Pures et Appliquèes, to appear.

[4] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers-Dordrecht, 1988.

[5] A. Jacquemard, M.A. Teixeira and D.J. Tonon, *Stability conditions in piecewise smooth dynamical systems at a two-fold singularity*, Journal of Dynamical and Control Systems, vol. **19**, (2013), 47-67.

[6] M.A. Teixeira, *Perturbation Theory for Non-smooth Systems*, Meyers: Encyclopedia of Complexity and Systems Science, vol. **152**, (2008).

DOI: 10.5540/03.2015.003.01.0035