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Modal analysis of an electromechanical system: a hybrid behavior

Roberta Lima[1], Rubens Sampaio[2]
Mechanical Engineering Department, PUC-Rio, Rio de Janeiro, RJ

Abstract. Electromechanical systems are composed by two interacting subsystems, a mechanical and an electromagnetic. This paper discusses the oscillatory response of a linear electromechanical system. The objective of the paper is to show that the oscillatory response of the chosen electromechanical system is provoked by the mutual interaction between mechanical and an electromagnetic subsystems, and to compare this oscillatory response with the response of purely mechanical systems. Natural frequencies and normal modes, are computed for the electromechanical system. The computed parameters involve mechanical and electromagnetic variables, i.e., they are hybrid, a novelty in the literature. Hybrid model coordinates and frequency responses graphs are also discussed.
Keywords. Electromechanical systems, natural frequencies, normal modes, resonance

1 Introduction

Electromechanical systems are an interesting type of dynamical systems. They are composed by two interacting subsystems, a mechanical and an electromagnetic. To properly characterize the dynamics of an electromechanical system, it is not sufficient to characterize the dynamics of each subsystem independently, it is necessary to include in the mathematical model of the system dynamics the mutual influence between the two subsystems [2, 5, 7].

The state of an electromechanical system involves mechanical and electromagnetic variables, and this is reflected in the initial value problem (IVP) that gives the system dynamics. The initial value problem is composed by a set of differential equations and initial conditions with these two types of variables, as for example, positions, velocities, angles, currents, and charges [4, 6]. In the set, the mutual interaction between the mechanical and an electromagnetic subsystems does not appear as a functional relation. The mutual interaction varies with the state of the subsystems and, consequently, depends on initial conditions [1].

The dynamic behavior of an electromechanical system depends on this mutual interaction, i.e., phenomena present in the system response reflects this interplay between the mechanical and electromagnetic subsystems. In this paper, the focus is in a special phenomenon: oscillations. A linear electromechanical system composed by a DC motor connected to a rigid disc, a motor-disc system is analyzed. This system has the minimum number of elements necessary to be classified as an electromechanical system. It is a bare minimum to study oscillatory response of electromechanical systems and to make modal analysis. The choice to address the problem in this bare minimum system was to highlight the effect of the mutual interaction between the mechanical and electromagnetic subsystems in the oscillatory response. Besides, the system was chosen as simple as possible so that the analyses could be done analytically. Natural frequencies and normal modes are computed. Differently from purely mechanical systems [3, 9], here these parameters involve

[1]robertalima@puc-rio.br, ORCID 0000-0002-3533-3054
[2]rsampaio@puc-rio.br, ORCID 0000-0002-2697-0019

mechanical and electromagnetic variables, i.e., the computed natural frequencies and normal modes are hybrid, a novelty in the literature.

1.1 Electromechanical natural frequencies

The hybrid natural frequencies are the frequencies at which the electromechanical system responds when there is no external excitation acting over it, that is, when the system is free. In our case, this means no external torque acting over the disc or no external source voltage applied over the electric circuit of the DC motor. In this situation, the hybrid natural frequencies also represents the frequencies at which occurs the interplay of energies between the mechanical and the electromagnetic subsystems.

1.2 Electromechanical normal modes

The dynamics of an electromechanical system is usually parametrized with purely mechanical and an electromagnetic variables, as positions, velocities, angles, currents, and charges. Since these variables are native and intrinsic to the problem, they are the most natural variables to parametrize the system dynamics. With such kind of variables, the set of differential equations present in the IVP that characterizes the dynamics of an electromechanical system is a coupled set of equations. This means that the equations of the set can not be solved independently.

The choice to parametrize the dynamics with native and intrinsic variables, easier to understand and visualize, generates a coupled set of equations in the IVP. However, if the dynamics were parametrized with a special set of variables, obtained from the hybrid normal modes and called modal coordinates, the set of equations would become uncoupled. The hybrid normal modes forms a basis of a vector space that can be used to represent the system dynamics.

Writing the system dynamics in terms of the modal coordinates turns possible to compute the system response for external excitations. For the system analyzed in this paper, the focus is on the computation of the response for harmonic external excitations and graphs of frequency response. Since the motor-disc system is linear and conservative, when it is excited harmonically with frequency equal to the natural frequency of the system, resonance appears, i.e., an electromechanical resonance, another novelty of the paper.

This paper is organized as follows. Section 2 presents the dynamics of the motor-disc, i.e., the initial value problem (IVP), that describes the dynamics of the analyzed electromechanical system. The homogeneous solution of the IVP, i.e., the system response when there is no external excitation acting over it is computed in Sect. 3. In this section the hybrid natural frequencies and normal modes are also computed. The decoupling of the equations of the IVP that gives the system dynamics using modal coordinates is made in Sec. 4. In Sec. 5, it is presented the system response for harmonic external excitations and the resonance is discussed.

2 Dynamics of the electromechanical system

The electromechanical system analyzed in this paper is a DC motor connected to a disc as shown in Figure 1. The initial value problem to the system is given in Equation (1) [8]. Find $(\alpha,\ z)$ such that, for all $t > 0$,

$$\begin{aligned} l\ddot{z}(t) + r\ \dot{z}(t) + k_e\dot{\alpha}(t) &= \nu(t)\,, \\ j_m\ddot{\alpha}(t) + b_m\dot{\alpha}(t) - k_e\dot{z}(t) &= \tau(t)\,, \end{aligned} \tag{1}$$

with the initial conditions $\dot{\alpha}(0) = \theta_0$, $\alpha(0) = \alpha_0$, $\dot{z}(0) = c_0$ and $z(0) = z_0$. In these equations, t is the time, ν is the source voltage, z is the electric charge, $\dot{\alpha}$ is the angular speed of the disc, l is the

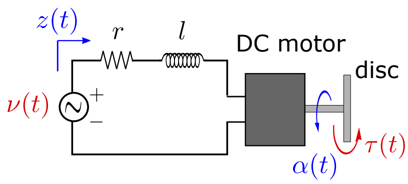


Figure 1: Electromechanical system.

electric inductance, j_m is the disc moment of inertia, b_m is the damping ratio in the transmission of the torque generated by the motor, k_e is the motor electromagnetic force constant, r is the electrical resistance, and τ is an external torque made over the disc.

The system state is given by four variables, two of them mechanical (angular velocity and position of the disc) and two of them electromagnetic (charge and current in the motor). These four variables are native and intrinsic to the problem, natural variables to parametrize the system state. The system dynamics, parametrized with these four variables, is given by an initial value problem comprising a set of two coupled differential equations. The coupling between the mechanical and electromagnetic subsystems is not given by a functional relation. It depends on the system state and, consequently, depends on initial conditions. Writing Equation (1) in matrix form, and assuming $b_m = 0$ and $r = 0$ to get a conservative system, it is obtained:

$$\begin{bmatrix} l & 0 \\ 0 & j_m \end{bmatrix} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\alpha}(t) \end{bmatrix} + \begin{bmatrix} 0 & k_e \\ -k_e & 0 \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ \dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} \nu(t) \\ \tau(t) \end{bmatrix}, \tag{2}$$

$$M\ddot{\mathbf{Y}}(t) + G\dot{\mathbf{Y}}(t) = \mathbf{F}(t)\,, \tag{3}$$

where M and G can be called as inertia and gyroscopic matrices respectively and $\mathbf{Y} = \begin{bmatrix} z \\ \alpha \end{bmatrix}$. The initial conditions become $\dot{\mathbf{Y}}(0) = \begin{bmatrix} c_0 \\ \theta_0 \end{bmatrix}$ and $\mathbf{Y}(0) = \begin{bmatrix} z_0 \\ \alpha_0 \end{bmatrix}$. Making $\dot{\alpha} = \theta$ and $\dot{z} = c$, where θ represents the angular velocity of the disc and c represents the current in the electric circuit of the DC motor, it is possible to rewrite Equation (2) as a system of first order differential equations given by

$$M\dot{\mathbf{X}}(t) + G\mathbf{X}(t) = \mathbf{F}(t)\,, \tag{4}$$

where $\mathbf{X} = \dot{\mathbf{Y}} = \begin{bmatrix} c \\ \theta \end{bmatrix}$. The initial conditions are given by $\mathbf{X}(0) = \begin{bmatrix} c_0 \\ \theta_0 \end{bmatrix}$. Since l and j_m are considered to be non-zero, Equation (4) can be rewritten as

$$\dot{\mathbf{X}}(t) = -M^{-1}\,G\,\mathbf{X}(t) + M^{-1}\,\mathbf{F}(t) = \dot{\mathbf{X}}(t) = A\mathbf{X}(t) + \mathbf{D}(t)\,, \tag{5}$$

where $A = -M^{-1}G$ and $\mathbf{D}(t) = M^{-1}\mathbf{F}(t) = \begin{bmatrix} \nu(t)/l \\ \tau(t)/j_m \end{bmatrix}$. The solution of Equation (5) is $\mathbf{X}(t) = \mathbf{X}_h(t) + \mathbf{X}_p(t)$, where X_h is the general solution of the associated homogeneous equation ($\dot{\mathbf{X}}_h = A\,\mathbf{X}_h$) and X_p is a particular solution of the non-homogeneous equation.

3 Homogeneous solution

It is proposed as solution to the associated homogeneous equation $\mathbf{X}_h = \mathbf{U}\,e^{\lambda\,t}$, where $\mathbf{U}$ is a non-zero constant vector and λ a scalar. Substituting the proposed general solution into the the associated homogeneous equation, it is obtained $(A - \lambda\,I)\mathbf{U} = \mathbf{0}$, which forms an eigenvalue problem.

3.1 Natural frequency and normal modes of the electromechanical system

Since $\mathbf{U} \neq \mathbf{0}$, the matrix $(A - \lambda I)$ is singular. Thus:

$$\det\Big(A - \lambda I\Big) = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k_e^2}{l\, j_m} = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm \frac{k_e}{\sqrt{l\, j_m}}\, i\,, \tag{6}$$

where $i = \sqrt{-1}$. Substituting the two eigenvalues $\lambda_{1,2}$ into the eigenvalue problem, it is possible to write $(A - \lambda_1 I)\mathbf{U}_1 = \mathbf{0}$ and $(A - \lambda_1 I)\mathbf{U}_2 = \mathbf{0}$. For $\lambda_1 = \frac{k_e}{\sqrt{l\, j_m}} i$, the associated eigenvector is $\mathbf{U}_1 = \begin{bmatrix} i\, j_m/\sqrt{l\, j_m} \\ 1 \end{bmatrix}$. For $\lambda_2 = -\frac{k_e}{\sqrt{l\, j_m}} i$, the associated eigenvector is $\mathbf{U}_2 = \begin{bmatrix} -i\, j_m/\sqrt{l\, j_m} \\ 1 \end{bmatrix}$. The eigenvalues $\lambda_{1,2}$ give a natural frequency of the system $\omega_n = \frac{k_e}{\sqrt{l\, j_m}}$. The eigenvectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are normal modes. Observe that the natural frequency, ω_n, and the normal modes are hybrid. They involve mechanical and electromagnetic parameters. Since two pairs of eigenvalues and eigenvectors were found, the general solution of the associated homogeneous equation should be a linear combination of the two found solutions:

$$\mathbf{X}_h(t) = a\, e^{\lambda_1 t}\, \mathbf{U}_1 + b\, e^{\lambda_2 t}\, \mathbf{U}_2 = \begin{bmatrix} \cos\left(\frac{k_e}{\sqrt{l\, j_m}} t\right) \frac{j_m}{\sqrt{l\, j_m}}\, h - \sin\left(\frac{k_e}{\sqrt{l\, j_m}} t\right) \frac{j_m}{\sqrt{l\, j_m}}\, d \\ \cos\left(\frac{k_e}{\sqrt{l\, j_m}} t\right) d + \sin\left(\frac{k_e}{\sqrt{l\, j_m}} t\right) h \end{bmatrix}, \tag{7}$$

where a and b are constants, $d = a + b$ and $h = i(a - b)$.

4 Decoupling the IVP that gives the system dynamics using the normal modes

With the eigenvalues and eigenvectors, spectral and modal matrices can be writing as:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{k_e}{\sqrt{l\, j_m}} i & 0 \\ 0 & -\frac{k_e}{\sqrt{l\, j_m}} i \end{bmatrix}, \quad P = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} = \begin{bmatrix} i j_m/\sqrt{l\, j_m} & -i j_m/\sqrt{l\, j_m} \\ 1 & 1 \end{bmatrix}. \tag{8}$$

It is possible to compute: $P^{-1} = \frac{1}{2\, i\, j_m/\sqrt{l\, j_m}} \begin{bmatrix} 1 & i j_m/\sqrt{l\, j_m} \\ -1 & i j_m/\sqrt{l\, j_m} \end{bmatrix}$. The matrix A can be written as $A = P\, \Lambda\, P^{-1}$. Since the focus of the paper is the computation of the response for harmonic external excitations, the source voltage ν is considered to be $\nu_0 \cos(\omega\, t)$ or $\nu_0 \sin(\omega\, t)$, and external torque τ of the form $\tau_0 \cos(\omega\, t)$ or $\tau_0 \sin(\omega\, t)$. Here, ν_0 and τ_0 represents the amplitudes of the external excitations and ω their frequencies. Calling by $\mathbf{X}_1$ and $\mathbf{X}_2$ the system states when the system is forced with the functions sine and cosine respectively, it possible to write:

$$\dot{\mathbf{X}}_1(t) = A\mathbf{X}_1(t) + \mathbf{D}_1(t) = A\mathbf{X}_1(t) + \begin{bmatrix} \nu_0/l \\ \tau_0/j_m \end{bmatrix} \cos(\omega\, t), \tag{9}$$

$$\dot{\mathbf{X}}_2(t) = A\mathbf{X}_2(t) + \mathbf{D}_2(t) = A\mathbf{X}_2(t) + \begin{bmatrix} \nu_0/l \\ \tau_0/j_m \end{bmatrix} \sin(\omega\, t). \tag{10}$$

Multiplying Equation (10) by the complex i and adding it to Equation (9) it is obtained:

$$\dot{\mathbf{S}}(t) = A\mathbf{S}(t) + \mathbf{B}(t) = A\mathbf{S}(t) + \begin{bmatrix} \nu_0/l \\ \tau_0/j_m \end{bmatrix} e^{i\,\omega\,t}, \tag{11}$$

where $\mathbf{S} = \mathbf{X}_1 + i\mathbf{X}_2$. Creating a new variable $\mathbf{S} = P\,\mathbf{Q}$, called modal variable, Equation (11) can be rewritten as

$$\begin{array}{rcccc} P\,\dot{\mathbf{Q}}(t) & = & A\,P\,\mathbf{Q}(t) & + & \mathbf{B}(t)\,, \\ P^{-1}\,P\,\dot{\mathbf{Q}}(t) & = & P^{-1}\,A\,P\,\mathbf{Q}(t) & + & P^{-1}\,\mathbf{B}(t)\,, \\ \dot{\mathbf{Q}}(t) & = & \Lambda\,\mathbf{Q}(t) & + & P^{-1}\,\mathbf{B}(t)\,, \end{array} \tag{12}$$

Thus, parametrizing the state of the electromechanical system with $\mathbf{Q}$, a modal variable with components q_1 and q_2, the equations that give the system dynamics become uncoupled and can be solved independently. They are:

$$\begin{cases} \dot{q}_1(t) - \dfrac{ik_e}{\sqrt{l\,j_m}} q_1(t) = \dfrac{1}{\dfrac{2\,i\,j_m}{\sqrt{l\,j_m}}} \left(\dfrac{\nu_0}{l} + \dfrac{i\,j_m}{\sqrt{l\,j_m}} \dfrac{\tau_0}{j_m} \right) e^{i\,\omega\,t}, \\ \dot{q}_2(t) + \dfrac{ik_e}{\sqrt{l\,j_m}} q_2(t) = \dfrac{1}{\dfrac{2\,i\,j_m}{\sqrt{l\,j_m}}} \left(-\dfrac{\nu_0}{l} + \dfrac{i\,j_m}{\sqrt{l\,j_m}} \dfrac{\tau_0}{j_m} \right) e^{i\,\omega\,t}. \end{cases} \tag{13}$$

5 Particular solution in terms of the modal coordinates

In this section of the paper, the particular solutions of each equation of Equation (13) is computed. These solutions are computed for two different cases. The first the case is when the system is harmonically excited at a frequency different from the natural frequency, ω_n. The second case is when it is excited at a frequency equal to ω_n.

5.1 External excitation at frequency different from the natural frequency of the system

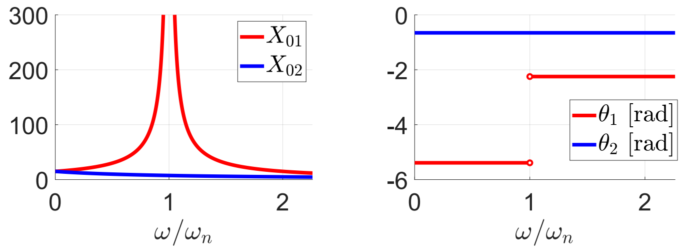
Considering that $\omega \neq \frac{k_e}{\sqrt{l\,j_m}}$, it is proposed as particular solution to the first differential of Equation (13) the expression $q_{1p}(t) = X_{01}\, e^{i(\omega\, t + \theta_1)}$, where X_{01} and θ_1 are constants to be determined. They represent, respectively, the amplitude of the proposed particular solution and the angular phase between the excitation and the particular solution. Substituting the proposed particular solution into the first equation of Equation (13) and, analyzing the modulus and phase of the complex terms of the obtained expression, it is possible to compute X_{01} and θ_1 as:

$$X_{01} = \frac{\left| \dfrac{1}{2} \sqrt{\left(\dfrac{\nu_0^2}{l\,j_m} + \dfrac{\tau_0^2}{j_m^2} \right)} \right|}{\left| \omega - \dfrac{k_e}{\sqrt{l\,j_m}} \right|}, \qquad \theta_1 = \begin{cases} \arctan\left(\dfrac{-\nu_0}{\sqrt{l\,j_m}} \dfrac{j_m}{\tau_0} \right) - \dfrac{3\pi}{2}, & \text{if } \omega < \dfrac{k_e}{\sqrt{l\,j_m}}\,, \\ \arctan\left(\dfrac{-\nu_0}{\sqrt{l\,j_m}} \dfrac{j_m}{\tau_0} \right) - \dfrac{\pi}{2}, & \text{if } \omega > \dfrac{k_e}{\sqrt{l\,j_m}}\,. \end{cases} \tag{14}$$

To second equation of Equation (13), it is proposed as particular solution the expression $q_{2p}(t) = X_{02}\, e^{i(\omega\, t + \theta_2)}$, where X_{02} and θ_2 are constants to be determined. They represent, respectively, the amplitude of the proposed particular solution and the angular phase between the excitation and the particular solution. Substituting the proposed particular solution into the second equation of Equation (13), and analyzing the modulus and phase of the complex terms of the obtained expression, X_{02} and θ_2 are computed:

$$X_{02} = \frac{\left|\frac{1}{2}\sqrt{\left(\frac{\nu_0^2}{l\, j_m} + \frac{\tau_0^2}{j_m^2}\right)}\right|}{\left|\omega + \frac{k_e}{\sqrt{l\, j_m}}\right|}, \qquad \theta_2 = \arctan\left(\frac{\nu_0}{\sqrt{l\, j_m}}\frac{j_m}{\tau_0}\right) - \frac{\pi}{2}. \tag{15}$$

Figures 2(a) and 2(b) show the frequency responses and phase graphs for the particular solutions of equations given in Equations (13). To plot the graphs, it was considered the following values to the system parameters: $l = 1.880 \times 10^{-4}$ H, $j_m = 1.210 \times 10^{-4}$ kg m^2, $\nu_0 = 1.000$ V, $k_e = 5.330 \times 10^{-2}$ V/(rad/s) and $\tau_0 = 1.000$ Nm. The motor parameters were obtained from the specifications of the motor Maxon DC brushless number 411678.



(a) (b)

Figure 2: (a) Frequency response graphs and (b) phase graphs for the particular solutions of Equations (13).

5.2 External excitation at frequency equal to the natural frequency of the system

Considering that $\omega = \omega_n = \frac{k_e}{\sqrt{l\, j_m}}$, resonance occurs for the first equation of Equation (13). Thus, the particular solution of this equation is not periodic and its amplitude grows linearly over time. This particular solution can be written as $q_{1pr}(t) = X_{01r}\, t\, e^{i(\omega t + \theta_{1r})}$, where X_{01r} and θ_{1r} are constants to be determined. Substituting it into the first equation of Equation (13) and analyzing the modulus and phase of the complex terms of the obtained expression, it is possible to compute X_{01r} and θ_{1r}:

$$X_{01r} = \frac{1}{2}\sqrt{\left(\frac{\nu_0^2}{l\, j_m} + \frac{\tau_0^2}{j_m^2}\right)}, \qquad \theta_{1r} = \arctan\left(\frac{-\nu_0}{\sqrt{l\, j_m}}\frac{j_m}{\tau_0}\right). \tag{16}$$

For the second equation of Equation (13), the proposed particular solution is $q_{2pr}(t) = X_{02r}\, e^{i(\omega t + \theta_{2r})}$, where X_{02r} and θ_{2r} are constants to be determined. Substituting it into the second equation of Equation (13) and analyzing the modulus and phase of the complex terms obtained, it is possible to write:

$$X_{02r} = \frac{\left|\frac{1}{2}\sqrt{\left(\frac{\nu_0^2}{l\, j_m} + \frac{\tau_0^2}{j_m^2}\right)}\right|}{\left|\omega + \frac{k_e}{\sqrt{l\, j_m}}\right|}, \qquad \theta_{2r} = \arctan\left(\frac{\nu_0}{\sqrt{l\, j_m}}\frac{j_m}{\tau_0}\right) - \frac{\pi}{2}. \tag{17}$$

6 Conclusions

In this paper, the oscillatory response of a simple and linear electromechanical system was analyzed. It was shown that the oscillatory response of the chosen electromechanical system is provoked by the mutual interaction between mechanical and an electromagnetic subsystems. Natural frequencies and normal modes, were computed. Since they involve mechanical and electromagnetic variables, they are hybrid. The hybrid natural frequency is the frequency at which occurs the interplay of energies between the mechanical and the electromagnetic subsystems. The hybrid normal modes forms a basis of a vector space that can be used to represent and decouple the system dynamics.

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