Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 9, n. 1, 2022.

Trabalho apresentado no XLI CNMAC, Unicamp - Campinas - SP, 2022.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics
Preprint

Exponential Stability Results for a mixture of elastic solids with frictional dissipation

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Abstract. We consider a mathematical model that describes a mixture of n elastic materials. We study the mixture problem with frictional damping. The problem consists of a linear system of n coupled hyperbolic partial differential equations. An existence and uniqueness result and an energy decay property are mentioned.

Palavras-chave. C_0 Semigroup, Exponential stability, strong stability, mixture of solids

1 Introduction

The theory of mixtures of solids has been widely investigated in the last decades. In recent years, an increasing interest has been directed to the study of the qualitative properties of solutions related to mixtures composed of two interacting continua. Several results concerning existence, uniqueness, continuous dependence and asymptotic stability can be found in the literature [2],[4], [8], [1]. In [8] R. Quintanilla made a full characterization of the asymptotic behavior of the following mixture model

$$\begin{aligned}\rho_1 u^1_{tt} &= a_{11}u^1_{xx} + a_{12}u^2_{xx} - \xi_0(u^1 - u^2) - \xi(x)(u^1_t - u^2_t),\\ \rho_2 u^2_{tt} &= a_{12}u^1_{xx} + a_{22}u^2_{xx} + \xi_0(u^1 - u^2) + \xi(x)(u^1_t - u^2_t).\end{aligned}$$

He proved the exponential decay of solutions when

$$\frac{a_{11}+a_{12}}{\rho_1} \neq \frac{a_{12}+a_{22}}{\rho_2} \tag{1}$$

and the coefficient of the relative velocity ξ can be localized. We mean that ξ is a non-negative function (which can be zero in a part of the domain) but

$$\int_0^l \xi(x)dx > 0.$$

We consider the coefficients defined in **[19]** for defined the following matrices, $\mathbf{R} = (\delta_{ij}\rho_i)$ is a diagonal positive definite matrix, $\mathbf{A} = (a_{ij})$ is a positive definite (real) symmetric matrix, $\mathbf{B} = (\xi(x)(-1)^{i+j})$ and $\mathbf{N} = (\xi_0(-1)^{i+j})$ are semipositive definite (real) symmetric matrix. Using these notations we can write the general case (system of n equations) as

$$\mathbf{R}U_{tt} = \mathbf{A}U_{xx} - \mathbf{N}U - \mathbf{B}(x)U_t. \tag{2}$$

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Note that if $\mathbf{C} = [1 \ -1]$ then $\mathbf{B} = \mathbf{C}^T \xi(x)\mathbf{C}$ and $\mathbf{N} = \mathbf{C}^T \xi_0 \mathbf{C}$. Moreover, the condition (1) is equivalent to

$$\text{rank } \left[\mathbf{C}, \ \mathbf{C}\mathbf{R}^{-1}\mathbf{A} \right] = 2.$$

Here we deal with the theory of elastic mixtures. We study the one dimensional model of a mixture of n interacting continua with reference configuration over $[0, l]$. The displacements of the particles of the continua at time t are denote by $u^1 := u^1(x_1, t), \ u^2 := u^2(x_2, t), \ \dots, u^n := u^n(x_n, t)$ where $x_i \in [0, l]$. We assume that the particles under consideration occupy the same position at time $t = 0$, so that $x = x_i$, in this theory the basic equations are

$$\mathbf{R}U_{tt} - \mathbf{A}U_{xx} + \mathbf{N}U + \mathbf{B}(x)U_t = 0. \tag{3}$$

with $U = (u^1, \cdots, u^n)$ and

$$\mathbf{R} = (\rho_i \delta_{ij})_{n\times n}, \quad \mathbf{A} = (a_{ij})_{n\times n}, \quad \mathbf{C} = (c_{ij})_{J\times n}, \quad \mathbf{B} = \mathbf{C}^T\mathbf{D}(x)\mathbf{C}, \quad \mathbf{N} = \mathbf{C}^T\mathbf{D}_0\mathbf{C}$$

$$\mathbf{D}(x) = (d_{ij}(x))_{J\times J}, \quad \mathbf{D}_0 = (d_{ij})_{J\times J}$$

where δ_{ij} is the Kronecker's delta, $\mathbf{A}$, $\mathbf{D}_0$ are positive definite (real) symmetric matrices and $\lambda_1(x) = \min\{\lambda \in \sigma(\mathbf{D}(x))\}$ is non-negative function of class C^1 (which can be zero in a part of the domain) but

$$\frac{1}{l}\int_0^l \lambda_1(x)dx > 0.$$

The initial conditions are given by

$$U(x, 0) = U_0(x), \quad U_t(x, 0) = U_1(x). \tag{4}$$

Finally, we consider Dirichlet boundary conditions

$$U(0, t) = U(l, t) = 0, \ , \quad t \in \mathbb{R}^+. \tag{5}$$

Therefore the dissipative mechanism is reflected by the rank of the matrix $\mathbf{B}$. If rank $\mathbf{B} = n$ (fully dissipative) then the solution semigroup is exponentially stable for all values of the structural parameters. On the other hand, when rank $\mathbf{B} = 0$, system (3) is conservative. Here the question is, what happen in case of

$$0 < \text{rank } \mathbf{B}(x) < n.$$

Is it possible that the above system is exponentially stable? or polynomially stable? or there exists oscillating solutions?. To give a complete answer the matrices: $\mathbf{W} = \mathbf{R}^{-1}\mathbf{A}$ and $\mathbf{C}$ will play an important role. Note that

$$\mathbf{D}(x) \text{ is a positive definite matrix } \implies \text{rank } \mathbf{B}(x) = \text{rank } \mathbf{C}.$$

The main result of this paper is to show that the semigroup associated to (3)-(5) is exponentially stable if and only if

$$\dim \text{ spam}\{\mathbf{C}_{(j)}\mathbf{W}^m; \ m = 0, 1, \cdots n-1\} = n, \tag{6}$$

where $\mathbf{C}_{(j)}$ is a row vector of $\mathbf{C}$. Moreover we prove that the above system never is polynomially stable. That is, we show that, if the system is not exponentially stable, then there exists oscillating solutions, (see Theorem 3.1). In particular our result implies in the corresponding semigroup is exponential stable if and only if it is strongly stable (as in the finite dimensional case). We believe that this property holds because the system has second order and zero order coupling terms with the same matrix structure given by $\mathbf{C}$. In [1] they full caracterization the case constant of (3)-(5).

This paper is organized as follows. In section 2 we establish the well posedness of the system. Finally, in Section 3 we prove that the system has the exponential stability property.

2 Semigroup formulation

From now on we use the semigroup theory to show the well posedness as well as the asymptotic properties. To do that let us introduce the phase space $\mathcal{H}$

$$\mathcal{H} = [H_0^1(0,l)]^n \times [L^2(0,l)]^n,$$

that is a Hilbert space with the norm

$$||(U,V)||^2_{\mathcal{H}} = \int_0^l U_x^* \mathbf{A} U_x dx + \int_0^l V^* \mathbf{R} V dx + \int_0^l U^* \mathbf{N} U dx$$

Let us introduce the operator $\mathcal{A}$ given by

$$\mathcal{A}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} V \\ \mathbf{R}^{-1}\mathbf{A}U_{xx} - \mathbf{R}^{-1}\mathbf{N}U - \mathbf{R}^{-1}\mathbf{B}V \end{pmatrix} \tag{7}$$

with domain

$$D(\mathcal{A}) = [H_0^1(0,l) \cap H^2(0,l)]^n \times [H_0^1(0,l)]^n.$$

Under this conditions the initial-boundary value problem can be rewritten as

$$\frac{d}{dt}\mathbf{U} = \mathcal{A}\mathbf{U}, \quad \mathbf{U}(0) = \mathbf{U}_0$$

where $\mathbf{U}(t) = (U(t), V(t))^\top$ and $\mathbf{U}_0 = (U_0, U_1)^\top$.

Theorem 2.1. *The operator $\mathcal{A}$ is the infinitesimal generator of a contractions C_0-semigroup, we denote as $\mathcal{S}_\mathcal{A}(t) = e^{\mathcal{A}t}$.*

Proof. Note that $D(\mathcal{A})$ is dense in $\mathcal{H}$ and a dissipative operator $\mathcal{A}$, that is

$$Re\,(\mathcal{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}} = -\int_0^l V^* \mathbf{B} V dx \le 0. \tag{8}$$

Therefore we only need to show that $0 \in \rho(\mathcal{A})$ (see Liu and Zheng [5] or Pazy [6]). In fact, we prove that for any $\mathbf{F} = (F, G) \in \mathcal{H}$ there exists a unique $\mathbf{U} = (U, V)$ in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{A}\mathbf{U} = \mathbf{F}$. In term of their components

$$V = F, \quad \mathbf{A}U_{xx} - \mathbf{N}U - \mathbf{B}(x)V = \mathbf{R}G \tag{9}$$

the above problem reduces to find $U \in [H^2 \cap H_0^1]^n$ such that

$$\mathbf{A}U_{xx} - \mathbf{N}U = \mathbf{B}(x)F + \mathbf{R}G.$$

But this problem is well posed and $\|\mathbf{U}\|_{\mathcal{H}} \le C\|\mathbf{F}\|_{\mathcal{H}}$, so $0 \in \varrho(\mathcal{A})$.

□

Other important tool we use is the characterization of the exponential stability of a C_0 semigroup was obtained by Huang [3], and Pruss [7] independently. Here we use the version due to Pruss.

Theorem 2.2. *Let $\mathcal{S}_\mathcal{A}(t)$ be a C_0-semigroup of contractions of linear operators on Hilbert space $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. Then $\mathcal{S}_\mathcal{A}(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{10}$$

and

$$\limsup_{|\lambda|\to+\infty} ||(i\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty \tag{11}$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of continuous linear functions in $\mathcal{H}$.

3 On the Stability of Semigroup

Note that $\mathcal{A}^{-1}$ is compact so $D(\mathcal{A})$ is compactly embedded into $\mathcal{H}$. Thus we conclude that the spectrum of the operator $\mathcal{A}$ consists entirely of isolated eigenvalues.

The resolvent equation can be written as

$$i\lambda\mathbf{U} - \mathcal{A}\mathbf{U} = \mathbf{F}. \tag{12}$$

where $\mathbf{U} = (U, V) \in D(\mathcal{A})$, $\mathbf{F} = (F, G) \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. Taking the inner product in $\mathcal{H}$ and considering the real part we get

$$\int_0^l V^*\mathbf{B}V\,dx = \mathrm{Re}\,(\mathbf{U}, \mathbf{F})_{\mathcal{H}}. \tag{13}$$

In terms of the components the resolvent equation (12) can be written as

$$i\lambda U = V + F \tag{14}$$

$$i\lambda V = \underbrace{\mathbf{R}^{-1}\mathbf{A}}_{=\mathbf{W}}U_{xx} - \mathbf{R}^{-1}\underbrace{\mathbf{C}^T\mathbf{D}_0\mathbf{C}}_{=\mathbf{N}}U - \mathbf{R}^{-1}\underbrace{\mathbf{C}^T\mathbf{D}(x)\mathbf{C}}_{=\mathbf{B}(x)}V + G. \tag{15}$$

The next Lemma will play an important role in the sequel.

Lemma 3.1. *The operator $\mathcal{A}$ satisfies the condition (10) if and only if (6) holds.*

Proof. Let us suppose that (10) is false, then there exists $0 \neq \mathbf{U} \in D(\mathcal{A})$ satisfying (12) with $\mathbf{F} = 0$. Using (13) we get

$$\int_0^l V^*\mathbf{B}V\,dx = 0 \quad \Longrightarrow \quad V^*(x)\mathbf{B}(x)V(x) = 0\ , \quad \text{almost always in } \ (0, l). \tag{16}$$

We obtained that

$$\mathbf{C}U = 0\ . \tag{17}$$

Now, (14)-(15) is equivalent to

$$-\lambda^2 U = \mathbf{W}U_{xx}, \quad \mathbf{C}U = 0. \tag{18}$$

So, we have

$$\mathbf{C}\mathbf{W}U_{xx} = 0 \quad \Rightarrow \quad \mathbf{C}\mathbf{W}U = 0.$$

multiplying by $\mathbf{CW}$ the first equation in (18) we get $\mathbf{C}\mathbf{W}^2U = 0$. Using induction we get that $\mathbf{C}\mathbf{W}^mU = 0$ for all m. If (6) holds then $U = 0$ which is a contradiction, therefore (6) is not true. To prove the other implication, note first that, if $Y_\star \in \mathbb{R}^n\backslash\{0\}$ is such that

$$\mathbf{C}Y_\star = 0, \quad (\mathbf{W} - \tau I)\,Y_\star = 0; \quad \tau > 0 \tag{19}$$

then the functions

$$\mathbf{U}_\nu = \Big(\sin(\frac{\nu\pi}{l}x)Y_\star\ ,\ i\lambda_\nu\sin(\frac{\nu\pi}{l}x)Y_\star\Big) \in D(\mathcal{A}),\ \nu \in \mathbb{N}, \tag{20}$$

are the eigenvectors of $\mathcal{A}$ with $i\lambda_\nu = i\frac{\nu\pi}{l}\sqrt{\tau}$ the corresponding imaginary eigenvalues, for $\nu \in \mathbb{N}$. Therefore $i\mathbb{R} \cap \sigma(\mathcal{A}) \neq \emptyset$. In fact, suppose that (6) is false. Then there exists $Y \neq 0$ such that

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{CW} \\ \vdots \\ \mathbf{CW}^{n-1} \end{bmatrix} Y \triangleq \begin{bmatrix} \mathbf{C}Y \\ \mathbf{CW}Y \\ \vdots \\ \mathbf{CW}^{n-1}Y \end{bmatrix} = \mathbf{0}$$

Which means that $\mathbf{CW}^m Y = 0$ for $m \in \mathbb{N} \cup \{0\}$. In particular

$$\mathbf{C}p(\mathbf{W})Y = 0, \quad \forall p \in \mathbb{R}[s]. \tag{21}$$

Let us denote by q the minimal polynomial of $\mathbf{W} = \mathbf{R}^{-1}\mathbf{A}$, therefore q can be written as

$$q(s) = (s - \tau_1)(s - \tau_2)...(s - \tau_m) \ , \quad q(\mathbf{W}) = 0.$$

Since $\mathbf{A} \succ 0$, $\mathbf{R}^{-1} \succ 0$, implies that τ_i are positive eigenvalues of $\mathbf{W}$.
Suppose that deg $q(s) = 1$. Then we have $(\mathbf{W} - \tau_1 I)Y = 0$. Taking $Y_\star := Y$ we see that $Y_\star$ solves system (19) and therefore our conclusion follows.

If $q(s) = (s-\tau_1)(s-\tau_2)$ then $(\mathbf{W}-\tau_1 I)(\mathbf{W}-\tau_2 I)Y = 0$. Here we can assume that $(\mathbf{W}-\tau_2 I)Y \neq 0$ otherwise we are in the previous case. Then $Y_\star = (\mathbf{W} - \tau_2 I)Y$ solves the system (19) for $\tau = \tau_1$.

Using induction, for deg $q(s) = m$ that there exists $q_0(s)$ divisor of the $q(s)$ such that $Y_\star := q_0(\mathbf{W})Y \neq 0$ solves system (19) for some $\tau_{i_0} > 0$. So $i\lambda_\nu = i\frac{\nu\pi}{l}\sqrt{\tau_{i_0}}$ is an eigenvalue for any $\nu \in \mathbb{N}$. Therefore there exist infinity many imaginary eigenvalues of the operator $\mathcal{A}$.

□

Finally, we prove the equivalence between exponential and strong stability. This in particular implies that the semigroup is never polynomially stable. Note that (13) implies

$$\int_0^l |\mathbf{C}V|^2 \lambda_1(x)dx \leq C||\mathbf{U}||_{\mathcal{H}}||\mathbf{F}||_{\mathcal{H}} \tag{22}$$

where $\lambda_1(x)$ is the first eigenvalue of $\mathbf{D}(x)$.

Lemma 3.2. *The operator $\mathcal{A}$ satisfies the condition (11) if (6) is holds.*

Proof. Suppose that (6) is not true. There then exists a sequence ω_ν with $\omega_\nu \to \infty$ and a sequence vectors functions $\mathbf{U}_\nu = (U_\nu, V_\nu) \in D(\mathcal{A})$ with unit norm in $\mathcal{H}$ such that as $\nu \to \infty$,

$$(i\omega_\nu I - \mathcal{A})\,\mathbf{U}_\nu \longrightarrow 0, \quad \text{in } \mathcal{H},$$

This is

$$F_\nu = i\omega_\nu U_\nu - V_\nu \to 0 \text{ in } \left[H_0^1(0,l)\right]^n \tag{23}$$

$$\mathbf{R}G_\nu = i\omega_\nu \mathbf{R}V_\nu - \mathbf{A}U_{\nu,xx} + \mathbf{N}U_\nu + \mathbf{B}V_\nu \to 0 \text{ in } \left[L^2(0,l)\right]^n. \tag{24}$$

Taking the inner product of $(i\omega_\nu I - \mathcal{A})\,\mathbf{U}_\nu$ by $\mathbf{U}_\nu$ in $\mathcal{H}$ and using the estimate (23) yields that

$$\int_0^l V_\nu^* \mathbf{B} V_\nu \, dx \ \to \ 0 \quad \Longrightarrow \quad \mathbf{D}^{1/2}\mathbf{C}V_\nu \ \to \ 0 \text{ in } [L^2(0.l)]^n.$$

Since $\mathbf{B}^{1/2}(x) \in [\mathbf{L}^\infty(0,l)]^{n\times n}$ we obtain

$$\mathbf{B}V_\nu \ \to \ 0 \text{ in } [L^2(0.l)]^n. \tag{25}$$

On other hand, we can easily deduce from (24) that

$$-\omega_\nu^2 \mathbf{R} U_\nu - \mathbf{A} U_{\nu,xx} \;=\; i\omega_\nu \mathbf{R} F_\nu + \mathbf{R} G_\nu - \mathbf{N} U_\nu - \mathbf{B} V_\nu. \tag{26}$$

Multiplying equation (26) by $\mathbf{CR}^{-1}$ we get

$$-\omega_\nu^2 \mathbf{C} U_\nu - \mathbf{CW} U_{\nu,xx} \;=\; i\omega_\nu \mathbf{C} F_\nu + \mathbf{C} G_\nu - \mathbf{CR}^{-1}\mathbf{N} U_\nu - \mathbf{CR}^{-1}\mathbf{B} V_\nu \tag{27}$$

multiplying (27) by by $\lambda_1(x)\mathbf{CW}^m$ we get

$$\int_0^l |\mathbf{CW}^m V|^2 \lambda_1(x) \;+\; |\mathbf{CW}^m U_x|^2 \lambda_1(x)\; dx \longrightarrow 0\ ,\ \text{as } \nu \longrightarrow +\infty. \tag{28}$$

for $m = 0, ..., k-1$.

Let $q(x)$ be a real function C^1 wich will be chosen later. Let $\mathbf{Q} \succ 0$ a real matrix. Note that for $W \in H^1(0,l)$, we have

$$2\ \text{Re} \int_0^l q(x) W^* \mathbf{Q} W_x\; dx \;=\; q(l) W^*(l) \mathbf{Q} W(l) \;-\; q(0) W^*(0) \mathbf{Q} W(0) - \int_0^l q'(x) W^* \mathbf{Q} W\; dx. \tag{29}$$

Taking the inner product of (26) with $q(x)U^*_{\nu,x}$, integrating by parts , we obtain

$$-\int_0^l q(x)[\omega_\nu^2 U^*_{\nu,x} \mathbf{R} U_\nu + U^*_{\nu,x} \mathbf{A} U_{\nu,xx}]\; dx \;=\; \int_0^l [i\omega_\nu U_\nu]^* [\mathbf{R} F_\nu q(x)]_x\; dx$$

$$+ \int_0^l q(x) U^*_{\nu,x} \mathbf{R} G_\nu\; dx - \int_0^l q(x) U^*_{\nu,x} \mathbf{N} U_\nu\; dx - \int_0^l q(x) U^*_{\nu,x} \mathbf{B} V_\nu\; dx,$$

and using (29), we obtain that

$$\int_0^l q'(x)[\omega_\nu^2 U^*_\nu \mathbf{R} U_\nu + U^*_{\nu,x} \mathbf{A} U_{\nu,x}]\; dx - q(x) U^*_{\nu,x} \mathbf{A} U_{\nu,x}|_0^l \;= \tag{30}$$

$$\begin{aligned} &2\text{Re} \int_0^l [i\omega_\nu U_\nu]^* [\mathbf{R} F_\nu q(x)]_x\; dx + \; 2\text{Re} \int_0^l q(x) U^*_{\nu,x} \mathbf{R} G_\nu\; dx \\ &- \;2\text{Re} \int_0^l q(x) U^*_{\nu,x} \mathbf{N} U_\nu\; dx - 2\text{Re} \int_0^l q(x) U^*_{\nu,x} \mathbf{B} V_\nu\; dx. \end{aligned}$$

Since $U_{\nu,x}$ and $\omega_\nu U_\nu$ are uniformly bounded in $[L^2(0,l)]^n$, the terms on the right hand side of (30) converge to zero.

Taking $q(x) = x$ we deduce from (30) and $||(U_\nu, V_\nu)||^2_{\mathcal{H}} = 1$ that

$$U^*_{\nu,x}(l) \mathbf{A} U_{\nu,x}(l) \longrightarrow \frac{1}{l}\ ;\ \ \text{as}\ \ \nu \longrightarrow \infty. \tag{31}$$

Taking $q(x) = l - x$ we deduce from (30) and $||(U_\nu, V_\nu)||^2_{\mathcal{H}} = 1$ that

$$U^*_{\nu,x}(0) \mathbf{A} U_{\nu,x}(0) \longrightarrow \frac{1}{l}\ ;\ \ \text{as}\ \ \nu \longrightarrow \infty. \tag{32}$$

We now take $q(x) = \int_0^x \lambda(s)\ ds$ in (30) to obtain that

$$\int_0^l [V_\nu^* \mathbf{R} V_\nu + U_{\nu,x}^* \mathbf{A} U_{\nu,x}] \lambda_1(x)\ dx \longrightarrow\ \lambda_0 = \frac{1}{l} \int_0^l \lambda_1(x)\ dx > 0\ . \tag{33}$$

this is contradictory with (28). Then (6) is true.

□

As a consequence of the above result we have that

Theorem 3.1. *Let us denote by $\mathcal{W} = \mathbf{R}^{-1}\mathbf{A}$ and let $\mathbf{D}(x)$ be a positive definite matrix in (a, b) with $0 < a < b < l$, $D \equiv 0$ in $(0, l) \setminus (a, b)$ and $\mathbf{B}(x) = \mathbf{C}^T D(x)\mathbf{C}$, then the following statements are equivalents.*

- *$\mathcal{S}_\mathcal{A}(t)$ is exponentially stable.*
- *$\mathcal{S}_\mathcal{A}(t)$ is strongly stable.*
- *Denoting by $\mathbf{C}_j$ the j-row vector of $\mathbf{C}$ then*

$$\dim \operatorname{span} \left\{ \mathbf{C}_j,\ \mathbf{C}_j\mathcal{W},\ \mathbf{C}_j\mathcal{W}^2,\ ...,\ \mathbf{C}_j\mathcal{W}^{n-1},\ j = 1, 2..., n \right\} = n. \tag{34}$$

4 conclusions

The Lemma 3.1 show that the corresponding semigroup is strongly stable (the imaginary axis is contained in the resolvent set of the infinitesimal generator) if and only if (6) is true. Moreover, this result is important for the demonstration of the Theorem 3.1. Moreover, The Theorem 3.1 implies the lack of polynomial stability to the corresponding semigroup.

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DOI: 10.5540/03.2022.009.01.0261