Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 9, n. 1, 2022.

Trabalho apresentado no XLI CNMAC, Unicamp - Campinas - SP, 2022.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics
Preprint

A note on Lie point symmetry to generalized fractional Burgers' equation

Junior Cesar Alves Soares[1]
FACET/UNEMAT, Barra do Bugres, MT
Felix Costa Silva [2]
FACET/UEMA, São Luis, MA
Stefânia Jarosz[3]
DMAT/UNICAMP, Campinas, SP
Gastão Silves Ferreira Frederico[4]
DMAT/UFC, Fortaleza, CE

Resumo. In this article we will present the method developed by [5] to find Lie symmetries for fractional differential equations. In particular, we will find the symmetries for the generalized Burgers equation.

Palavras-chave. Lie Symmetry, Fractional Calculus, Reimann-Liouville

1 Introduction

The Lie symmetries method is a very useful tool for finding exact solutions of partial differential equations (PDEs), as it has the property of reducing the order of an PDEs or even reduce it to an ordinary differential equation (ODE). Finding the Lie symmetries of an PDE can help to better understand the solutions and even be the starting point for others. In the same way it is possible to apply this method to the fractional partial differential equations for the same purposes indicated in the case of integer order [7].

Consider an FPDE of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F\left[U\right], \tag{1}$$

where $u(x,t)$ denotes the unknown function, and $F\left[U\right] = F(x,t,u,u_x,u_{xx},u_{xt},\ldots)$ is a known function. In this case $0 < \alpha \leq 1$ and the derivative considered is in the Riemann-Liouville sense. Let us assume that the FPDE is invariant under ϵ, a continuous transformation parameter. So we can write

[1]juniorcasoares@unemat.br
[2]felix@cecen.uema.br
[3]stjarosz@gmail.com
[4]gastao.frederico@ua.pt

$$
\begin{aligned}
&\bar{t} = t + \epsilon\tau(x,t,u) + \mathcal{O}(\epsilon^2),\\
&\bar{x} = x + \epsilon\xi(x,t,u) + \mathcal{O}(\epsilon^2)\\
&\bar{u} = u + \epsilon\eta(x,t,u) + \mathcal{O}(\epsilon^2)\\
&\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta_\alpha^0 + \mathcal{O}(\epsilon^2)\\
&\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial u}{\partial x} + \epsilon\eta_x^{(1)} + \mathcal{O}(\epsilon^2)\\
&\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = \frac{\partial^2 u}{\partial x^2} + \epsilon\eta_{xx}^{(2)} + \mathcal{O}(\epsilon^2)\\
&\vdots
\end{aligned}
\tag{2}
$$

where $\tau(x,t,u)$, $\xi(x,t,u)$ e $\eta(x,t,u)$ are infinitesimal coefficients and

$$
\begin{aligned}
\eta_x^{(1)} &= \eta_x + (\eta_u - \xi_x)\,u_x - \tau_x u_t - \xi_u (u_x)^2 - \tau_u u_x u_t;\\
\eta_{xx}^{(2)} &= \eta_{xx} + (2\eta_{xx} {-}_{xx})\,u_x - \tau_{xx} u_t + (\eta_u - 2\xi_x)\,u_{xx} - 2\tau_x u_{xt} + (\eta_{uu} - 2\xi_{xu})\,(u_x)^2\\
&\quad - 2\tau_{xu} u_x u_t - \xi_{uu}(u_x)^3 - \tau_{uu}(u_x)^2 u_t - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}.
\end{aligned}
\tag{3}
$$

are extended infinitesimal coefficients of order $1,2$ respectively and η_α^0 is the extended infinitesimal coefficients of order α [10].

The infinitesimal generator associate to (1) is given by

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}.$$

Since the lower bound of the integral in the definition of $\dfrac{\partial^\alpha u}{\partial t^\alpha}$ is fixed and therefore must be invariant with respect to the transformations (2) we have the invariance condition

$$\tau(t,x,u)\Big|_{t=0} = 0.$$

As the equation (1) is fractional we have that the extended infinitesimal generator will be

$$X^\alpha = X + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_{xx}^{(1)}\frac{\partial}{\partial u_{xx}} + \eta_\alpha^{(0)}\frac{\partial}{\partial u^\alpha}, \tag{4}$$

where

$$\eta_\alpha^{(0)} = D_t^\alpha(\eta) + \underbrace{\xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x)}_{(1)} + \underbrace{D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u)}_{(2)}. \tag{5}$$

Using Leibniz's rule in the equation (5) we have:

1. For the expression in (1) we have

$$\begin{aligned}\xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) &= \xi D_t^\alpha(u_x) - \left[\sum_{n=0}^{\infty} D_t^{\alpha-n}(u_x) D_t^n(\xi)\right]\\ &= \xi D_t^\alpha(u_x) - \left[\xi D_t^\alpha(u_x) + \sum_{n=1}^{\infty} D_t^{\alpha-n}(u_x) D_t^n(\xi)\right]\\ &= -\sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u_x) D_t^n(\xi). \end{aligned} \tag{6}$$

2. For the expression in (2) we have

$$\begin{aligned}&D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u)\\ &= \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u) D_t^n(D_t(\tau)) - \left[\sum_{n=0}^{\infty}\binom{\alpha+1}{n} D_t^{\alpha+1-n}(u) D_t^n(\tau)\right] + \tau D_t^{\alpha+1}(u)\\ &= \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u) D_t^n(D_t(\tau)) - \left[\tau D_t^{\alpha+1}(u) + \sum_{n=1}^{\infty}\binom{\alpha+1}{n} D_t^{\alpha+1-n}(u) D_t^n(\tau)\right] + \tau D_t^{\alpha+1}(u)\\ &= \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u) D_t^{n+1}(\tau) - \left[\sum_{n=1}^{\infty}\binom{\alpha+1}{n} D_t^{\alpha+1-n}(u) D_t^n(\tau)\right].\end{aligned}$$

Now, in the previous expression in square brackets, we introduce the following change of variable $n := n+1$ like this

$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u) D_t^{n+1}(\tau) - \left[\sum_{n=0}^{\infty}\binom{\alpha+1}{n+1} D_t^{\alpha-n}(u) D_t^{n+1}(\tau)\right]\\ &= \sum_{n=0}^{\infty}\left[\binom{\alpha}{n} - \binom{\alpha+1}{n+1}\right] D_t^{\alpha-n}(u) D_t^{n+1}(\tau).\end{aligned}$$

Using Pascal's identity $\binom{\alpha}{n} + \binom{\alpha}{n+1} = \binom{\alpha+1}{n+1}$, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{\alpha}{n+1} D_t^{\alpha-n}(u) D_t^{n+1}(\tau)\\ &= \alpha D_t(\tau) D_t^\alpha(u) + \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{\alpha-n}(u) D_t^{n+1}(\tau).\end{aligned} \tag{7}$$

Substituting the equations (6) and (7) into the equation (5), we get

$$\eta_\alpha^{(0)} = D_t^\alpha(\eta) - \alpha D_t(\tau) D_t^\alpha(u) - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}(u_x) D_t^n(\xi) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{\alpha-n}(u) D_t^{n+1}(\tau). \tag{8}$$

We need to calculate the term $D_t^\alpha(\eta)$ from (8) remembering that $\eta = \eta(t,x,u)$ and $u = u(t,x)$, to do so, we will use Leibniz's rule and then the chain rule.

DOI: 10.5540/03.2022.009.01.0231

a) We can write $D_t^\alpha(\eta) = D_t^\alpha(1 \cdot \eta)$ and using Leibniz's rule we get

$$D_t^\alpha \left[1 \cdot \eta(t, x, u(t, x))\right] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}(1) D_t^n \left[\eta(t, x, u(t, x))\right].$$

b) Applying the chain rule to $D_t^n \left[\eta(t, x, u(t, x))\right]$ we get

$$D_t^n \left[\eta\right] = \sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{m} \sum_{r=0}^{k} \binom{k}{r} \frac{1}{k!} \left[-u(t, x)\right]^r D_t^m \left[(u(t, x))^{k-r}\right] \frac{\partial^{n-m+k}\eta(t, x, u)}{\partial t^{n-m} \partial u^k}.$$

Therefore, we obtain

$$\begin{aligned} D_t^n \left[\eta(t, x, u(t, x))\right] = &\sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{r=0}^{k} \binom{\alpha}{n} \binom{m}{n} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \times \\ &\times \left[-u(t, x)\right]^r D_t^m \left[(u(t, x))^{k-r}\right] \frac{\partial^{n-m+k}\eta(t, x, u)}{\partial t^{n-m} \partial u^k}. \end{aligned} \tag{9}$$

The equation (9) can be written by selecting the terms that have derivatives in u and are linear in u (See [5, 6] e [8]). Precisely, these terms are obtained when $k = 0$ and $k = 1$, in this way the expression can be rewritten as

$$D_t^\alpha(\eta) = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_u \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^n(\eta_u)}{\partial t^\alpha} D_t^{\alpha-n}(u) + \mu, \tag{10}$$

with

$$\begin{aligned} \mu = &\sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k} \binom{\alpha}{n} \binom{m}{n} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \times \\ &\times \left[-u(t, x)\right]^r D_t^m \left[(u(t, x))^{k-r}\right] \frac{\partial^{n-m+k}\eta(t, x, u)}{\partial t^{n-m} \partial u^k}. \end{aligned} \tag{11}$$

Remembering that the expression μ is equal to zero when η is linear in u or, equivalently, $\mu_{uu} = 0$.

Therefore, substituting the equation (10) in the equation (8) we find the α-th extended infinitesimal

$$\begin{aligned} \eta_\alpha^{(0)} = &\frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] \times \\ &\times D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x). \end{aligned} \tag{12}$$

Returning to the operator (4) we have that

$$X^\alpha \Delta_1 = 0 \quad \text{when} \quad \Delta_1 = 0,$$

with $\Delta_1 = \dfrac{\partial^\alpha u}{\partial t^\alpha} - F[U]$.

As an example we use this algorithm presented for the fractional Burgers equation.

DOI: 10.5540/03.2022.009.01.0231

2 Generalized Burgers' equation

The Burgers equation appeared around 1915, in a work by the physicist Bateman. Is it over there has been of great interest in physics, because of its statistical properties and because of its similarity to the 1-dimensional Navier-Stokes equations (see [1, 3]). The non-homogeneous form of the Burgers equation has many applications in the study of gas dynamics, acoustics, diffusion and advection phenomena. The original equation is given by:

$$u_t = \nu u_{xx} - u u_x. \tag{13}$$

In the paper [2] was considered by authors the Burgers' equation in the following form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} + u u_x, \tag{14}$$

where $0 < \alpha < 1$ from there they reduced the FPDE into an FODE via Lie symmetries.

In the work [11] was studied the Burgers equation given by:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -b u_{xx} - a u u_x, \tag{15}$$

where a, b are constants.

In the article [9] exact solutions were found for the inviscid Burgers' equation given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -f(u) u_x + g(u), \tag{16}$$

where f, g are smooth functions of u with $f \neq 0$.

3 Lie Symmetries to generalized Fractional Burgers' equation

In this work we consider the generalized fractional Burgers' equation, this formulation is inspired by the article [4] that makes a consistent study for the case where the time derivative is of integer order.

In the formulation made in this work, we use the equation in the sense of the fractional derivative as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} + e^{bu} u_x, \quad u = u(x,t)$$

where $0 < \alpha \leq 1$, $b = const. \neq 0$ and the Riemann-Liouville derivative.

$$\Delta_1 = \frac{\partial^\alpha u}{\partial t^\alpha} - u_{xx} - e^{bu} u_x$$

and by the invariance criterion we have:

$$X^\alpha \Delta_1 = X \Delta_1 + \eta_x^{(1)} \frac{\partial}{\partial u_x} \Delta_1 + \eta_{xx}^{(2)} \frac{\partial}{\partial u_{xx}} \Delta_1 + \eta_\alpha^{(0)} \Delta_1 = 0,$$

when $\Delta_1 = 0$.

So, we get

$$\eta b e^{bu} u_x - \eta_x^{(1)} e^{bu} - \eta_{xx}^{(1)} + \eta_\alpha^{(0)} = 0.$$

This equation depends on the variables $u_x, u_{xx}, u_{xt}, u_t, \cdots$ and $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$ to $n = 1, 2, 3, \cdots$ which are independent. Substituting the expressions of $\eta_x^{(1)}, \eta_{xx}^{(2)}, \eta_\alpha^{(0)}$ and separating the expressions into powers of u we get the following system:

$$\begin{cases} \xi_u = \xi_t = \tau_u = \tau_x = \eta_{uu} = 0, \\ \binom{\alpha}{n}\partial_t(\eta_u) - \binom{\alpha}{n+1}D_t^{n+1}(\tau) = 0, \text{ para } n = 1, 2, 3, \cdots, \\ \xi^{''}(x) - e^{bu}\alpha\tau^{'}(t) - 2\eta_{xu} + e^{bu}\xi^{'}(x) - \eta b e^{bu} = 0, \\ 2\xi^{'}(x) - \alpha\tau^{'}(t) = 0, \\ \partial_t^\alpha(\eta) - u\partial_t^\alpha - \eta_{xx} - e^{bu}\eta_x = 0. \end{cases}$$

Solving the previous system, we have

$$\xi = c_1x + c_2, \quad \tau = \frac{2c_1t}{\alpha}, \quad \eta = -\frac{c_1}{b},$$

where c_1 and c_2 are arbitrary constants. The infinitesimal generator is:

$$X = (c_1x + c_2)\frac{\partial}{\partial x} + \frac{2c_1t}{\alpha}\frac{\partial}{\partial t} - \frac{c_1}{b}\frac{\partial}{\partial u}.$$

Thus, we have that the associated Lie algebra is of dimension three, being the elements of the base given by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} - \frac{1}{b}\frac{\partial}{\partial u}.$$

We can use the infinitesimal generator X_2, for example, to find the variable transformation in order to reduce the fractional generalized Burgers equation to an ordinary fractional differential equation [8].

4 Discussion

The Lie symmetries approach is an excellent tool for finding exact solutions of partial differential equations and has been used to find solutions to the equations in their fractional versions. Since 2007, with the work of Gazizov et.al, many articles have used this technique. There are still many questions for the application of Lie Symmetries in Fractional Calculus, for example, if the technique is valid for all fractional operators and what are the infinitesimal generators for operators with non-singular kernels.

Furthermore, our purpose is to calculate Lie symmetries for several functions $g(u)$ that are smooth and to reduce the generalized Fractional Burgers equation for each case to an FODE. In this perspective, we have already performed the calculations of Lie symmetries in the sense of the Reimann-Liouville derivative and we are analyzing the cases.

Acknowledgments

The authors thank the universities UNEMAT, UEMA, Unicamp and UFC for their support in creating the research group.

References

[1] Harry Bateman. "Some recent researches on the motion of fluids". In: **Monthly Weather Review** 43.4 (1915), pp. 163–170.

[2] Bibekananda Bira, Tungala Raja Sekhar, and Dia Zeidan. "Exact solutions for some time-fractional evolution equations using Lie group theory". In: **Mathematical Methods in the Applied Sciences** 41.16 (2018), pp. 6717–6725.

[3] Mayur P Bonkile et al. "A systematic literature review of Burgers' equation with recent advances". In: **Pramana** 90.6 (2018), pp. 1–21.

[4] Igor L Freire. "Note on Lie point symmetries of Burgers Equations". In: **Trends in Computational and Applied Mathematics** 11.2 (2010), pp. 151–157.

[5] Raphael K Gazizov, AA Kasatkin, and S Yu Lukashchuk. "Continuous transformation groups of fractional differential equations". In: **Vestnik Usatu** 9.3 (2007), p. 21.

[6] RK Gazizov, AA Kasatkin, and S Yu Lukashchuk. "Group-invariant solutions of fractional differential equations". In: **Nonlinear Science and Complexity**. Springer, 2011, pp. 51–59.

[7] Mir Sajjad Hashemi and Dumitru Baleanu. **Lie symmetry analysis of fractional differential equations**. Chapman and Hall/CRC, 2020.

[8] R Sahadevan and T Bakkyaraj. "Invariant analysis of time fractional generalized Burgers and Korteweg–de Vries equations". In: **Journal of mathematical analysis and applications** 393.2 (2012), pp. 341–347.

[9] Muhammad Saqib et al. "On Lie symmetry analysis of nonhomogeneous generalized inviscid and fractional Burgers' equation". In: **Mathematical Methods in the Applied Sciences** 44.11 (2021), pp. 8726–8738.

[10] Junior C. A. Soares. "Cálculo Fracionário e as equações de evolução". PhD thesis. Unicamp, 2016.

[11] BA Tayyan and AH Sakka. "Symmetries and exact solutions of conformable fractional partial differential equations". In: **Palestine Journal of Mathematics** 9.1) (2020).

