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# Discontinuous Galerkin Finite Element Analysis of a Thermal coupling problem

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**Abstract**. In this work we use mixed formulations, based on the Raviart-Thomas (RTH) methods, considering the weak flow form between the elements of the finite element mesh. The thermal problem analyzed is an elliptical nonlinear problem. We propose a analysis, of model, considering the two elliptic coupled equations, in the mixed form and show numerical results which confirm optimal convergence rates for the flows in $H(\text{div};\Omega)$ and scalar variables in $L^2(\Omega)$.

**Key words**. Raviart-Thomas methods, Discontinuous Galerkin finite element, elliptical nonlinear problem, mixed methods,

## 1 Introduction

We considered the following termally coupled nonlinear elliptic problem

$$-\text{div}(\mu(\theta)\nabla u) = f, \text{ in } \Omega \tag{1a}$$

$$-\text{div}(\nabla\theta) = \mu(\theta)|\nabla u|^2, \text{ in } \Omega \tag{1b}$$

$$u = 0, \text{ on } \partial\Omega \tag{1c}$$

$$\theta = 0, \text{ on } \partial\Omega \tag{1d}$$

where $0 < \mu_1 < \mu(\theta) \leq \mu_2$ is eletrical condutibility, $\theta : \Omega \to I\!R$ is temperature and $u : \Omega \to I\!R$ eletrical potential. The open set $\Omega \subset I\!R^p$, $p = 2, 3$.

This problem was trated in [3, 6], where the domain $\Omega$ was supposed regular set, in the same sense as the definition in [4]. From this, the authors establish the existence, uniqueness and regularity of the solution, whith charge source $f$ satisfying $C\|f\|^2_{L^2(\Omega)} < 1$, where $C$ is a constant. We make the following hypothesis

$$\|\boldsymbol{\sigma}\|_{L^\infty(\Omega)} < \Gamma, \tag{2}$$

where $\boldsymbol{\sigma} = \mu(\theta)\nabla u$ and $\Gamma = \Gamma(\mu_1, \mu_2, \Omega, f)$ is a constant. It is hypothesis (2) simplifies mathematical analysis, so we focus on the numerical analysis of the discontinuous Galerkin method of the problem. Another hypothesis about regularity of function $\mu = \mu(s)$, $s \in I\!R$, whatever of Lipchitz, this is, there is a constant $L > 0$, such that

$$|\alpha(\theta_1) - \alpha(\theta_2)| \leq L\|\theta_1 - \theta_2\|_{L^2(\Omega)}, \quad \text{or } \|\mu'\|_{L^\infty(\Omega)} \leq L \tag{3}$$

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## 2 Mixed Problem

We consider the following problem in the mixed form, that regard fluxes $\boldsymbol{\sigma} = \mu(\theta)\nabla u$ and $\boldsymbol{q} = \nabla\theta$. Find $\boldsymbol{\sigma}, u, \boldsymbol{q}, \theta$, satisfying

$$\alpha(\theta)\boldsymbol{\sigma} = \nabla u \ \text{ in } \ \Omega \tag{4a}$$

$$-\text{div}\boldsymbol{\sigma} = f \ \text{ in } \ \Omega \tag{4b}$$

$$\boldsymbol{q} = \nabla\theta \,\text{in}\, \Omega \tag{4c}$$

$$-\text{div}(\boldsymbol{q}) = \alpha(\theta)|\boldsymbol{\sigma}|^2, \ \text{ in } \ \Omega \tag{4d}$$

$$u = 0 \text{ and } \ \theta = 0 \ \text{ on } \partial\Omega, \tag{4e}$$

where $\alpha(\theta) = \mu^{-1}(\theta)$ it's called electrical resistance, that is the inverse the electrical conductibility.

Considering the funcional spaces $\boldsymbol{U} = H(\text{div};\Omega) = \{\boldsymbol{\tau} \in (L^2(\Omega))^N : \nabla.\boldsymbol{\tau} \in L^2(\Omega)\}$ and $V = L^2(\Omega)$, we have the variacional form: find $(\boldsymbol{\sigma}, u)$ and $(\boldsymbol{q}, \theta) \in \boldsymbol{U} \times V$ that satisfy

$$(\alpha(\theta)\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\text{div}\boldsymbol{\tau}, u) = 0, \forall\, \boldsymbol{\tau} \in \boldsymbol{U} \tag{5a}$$

$$-(\text{div}\boldsymbol{\sigma}, v) = (f, v), \, \forall v \, \in V \tag{5b}$$

$$(\boldsymbol{q}, \boldsymbol{\tau}) - (\text{div}\boldsymbol{\tau}, \theta) = 0, \forall\, \boldsymbol{\tau} \in \boldsymbol{U} \tag{5c}$$

$$-(\text{div}\boldsymbol{q}, v) = (\alpha(\theta)|\boldsymbol{\sigma}|^2, v), \, \forall\, v \, \in V \cap L^\infty(\Omega) \tag{5d}$$

In (5d) there is a asymmetry in funcitons spaces, mamely $\alpha(\theta)|\boldsymbol{\sigma}|^2 \in L^1(\Omega)$ it makes $v \in L^\infty(\Omega)$, on the order hand, employing (2) and regularity $\|\boldsymbol{\sigma}\|_{L^2(\Omega)} \leq C_\Omega \|f\|_{L^2(\Omega)}$, for Diriclet problem (5b), we have

$$\begin{aligned} \left|\left(\alpha(\theta)|\boldsymbol{\sigma}|^2, v\right)\right| &\leq \mu_1^{-1}\|\boldsymbol{\sigma}\|_{L^\infty(\Omega)}\|\boldsymbol{\sigma}\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \\ &\mu_1^{-1}\Gamma C\|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}, \, \forall\, v \in L^2(\Omega) \end{aligned} \tag{6}$$

where $C = C(\Omega, \mu_1, \mu_2)$ is the constant, that depend of the domain. The inequality (6) show that the inner products in (5d) is equivalent the symmetrical form

$$(\text{div}\boldsymbol{q}, v) = (\alpha(\theta)|\boldsymbol{\sigma}|^2, v), \, \forall\, v \, \in L^2(\Omega). \tag{7}$$

Given $\theta$, it is easy to see that $(\alpha(\theta)\boldsymbol{\tau}, \boldsymbol{\tau}) \geq \frac{1}{\mu_2}\|\boldsymbol{\tau}\|^2_{\boldsymbol{U}}, \quad \forall \boldsymbol{\tau} \in \boldsymbol{U}$. Moreover, the LBB conditions holdis: there exists a constant $\beta > 0$ such that $\inf_{v\in V} \sup_{\boldsymbol{\tau}\in\boldsymbol{U}} (\nabla.\boldsymbol{\tau}, v) \geq \beta\|\boldsymbol{\tau}\|_{\boldsymbol{U}}\|v\|_V$ Hence, we have existence of solutions for (5a)-(5b) and (5c)-(5d), and still

$$\|\boldsymbol{\tau}\|_{\boldsymbol{U}} + \|u\|_V \leq C\|f\|_V \tag{8}$$

$$\|\boldsymbol{q}\|_{\boldsymbol{U}} + \|\theta\|_V \leq C\Gamma\|f\|_V \tag{9}$$

The existence and uniqueness of solution to problem (5a)-(5b) and (5c)-(5d) is establish in [1]

The results about the regularity of the solutions and the convergence of the fixed-point algorithm for the problem (10) , were demonstrated in the work [3].

**Point fixed algorithm**

we consider the problem in the e-nth iteration Find $(u^n, \theta^n) \in H^1_0(\Omega) \times H^1_0(\Omega)$, such that

$$(\mu(\theta^{n-1})\nabla u^n, \nabla v) = (f, v), \ \ \forall\, v \in \, H^1_0(\Omega) \tag{10a}$$

$$(\nabla\theta^n, \nabla\eta) = (\mu(\theta^{n-1})|\nabla u|^2, \eta), \quad \eta \in H^1_0(\Omega) \tag{10b}$$

**Theorem 2.1.** *There is convergence to solve the problem (10) for iteration over $n$, and*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^n\|_{L^2(\Omega)} + \|\nabla(u - u^n)\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}M(f)^{n-1}\|\nabla(\theta - \theta^0)\|_{L^2(\Omega)}, \tag{11}$$

$$\|\boldsymbol{q} - \boldsymbol{q}^n\|_{L^2(\Omega)} + \|\nabla(\theta - \theta^n)\|_{L^2(\Omega)} \leq M(f)^n\|\nabla(\theta - \theta^0)\|_{L^2(\Omega)}, \tag{12}$$

*where $M(f) = C\|f\|_{L^2(\Omega)} < 1$.*

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# 3 Hybridized Mixed Method (RTH)

In this section we will demonstrate error estimates for the approximations of the coupled problem via the Raviart -Thomas method [5], in the norms $H(\text{div};\Omega)$ and $L^2(\Omega)$ for the flow and the scalar variables, respectively. For this, we continue to consider $\Omega$, a polyhedral domain and $\partial\Omega_D = \partial\Omega$, where is the partition of $\Omega$ into elements denoted by $T$, satisfying the typical conditions for finite elements [2].

## 3.1 Raviart-Thomas Spaces

Sendo $h_T$ o diâmetro de $T$; denotamos $h = \max_{T\in\mathcal{T}_h} h_T$; $\mathcal{E}_h$ o conjunto das faces (ou lados) indicados por $E$ e $\partial T \in \partial\mathcal{T}_h$, isto é, o contorno do elemeto $T$ pertencente ao conjunto de todos os contornos em $\mathcal{T}_h$ e finalmente $\boldsymbol{\nu}$ o vetor normal a $\partial\Omega$.

We consider the Hilbert spaces in the discretized domain $\mathcal{T}_h$ with the set of edges $\partial\mathcal{T}_h$

$$L^2(\mathcal{T}_h) := \{v : v_h \in L^2(T),\, \forall \in \mathcal{T}_h\},\; L^2(\partial\mathcal{T}_h) := \{v : v_h \in L^2(\partial T),\, \forall\, T \in \mathcal{T}_h\}, \tag{13}$$

if $u, v \in L^2(\mathcal{T}_h)$ the inner products are $(u,v)_T := \int_T uvdx$, and $(u,v)_{\mathcal{T}_h} := \sum_T (u,v)_T$, and if $\lambda, m \in L^2(\partial\mathcal{T}_h)$ the inner products are $\langle\lambda, m\rangle_{\partial T} := \int_{\partial T}\lambda m ds$, and $\langle\lambda,m\rangle_{\partial\mathcal{T}_h} := \sum_T \langle\lambda,m\rangle_{\partial T}$, whith the norms $\|v\|_{L^2(\mathcal{T}_h)} = \sqrt{(v,v)_{\mathcal{T}_h}}$, $|v|_{L^2(\partial\mathcal{T}_h)} = \sqrt{\langle v, v_h\rangle_{\partial\mathcal{T}_h}}$ respectively. The Raviart-Thomas space of order $k$ is denoted $RT_k(T) := \mathcal{P}_k(T) \oplus \boldsymbol{x}.\mathcal{P}_k(T)$.

We defined the follows approximations spaces :

$$\boldsymbol{\Sigma}_h := \left\{\boldsymbol{\tau}_h \in (L^2(\Omega))^N : \boldsymbol{\tau}_h|_T \in RT_k(T),\, \forall T \in \mathcal{T}_h\right\}; \tag{14a}$$

$$\mathcal{V}_h := \{v_h \in L^2(\Omega) : v_h|_T \in \mathcal{P}_k(T),\, \forall T \in \mathcal{T}_h\}; \tag{14b}$$

$$\mathcal{M}_h := \{m_h \in L^2(\mathcal{E}_h) : m|_E \in \mathcal{P}_k(E), m|_E = 0,\, E \subset \partial\Omega,\, \forall\, E \in \mathcal{E}_h\}; \tag{14c}$$

$$\mathcal{W}_h := \boldsymbol{\Sigma}_h \times \mathcal{V}_h \times \mathcal{M}_h. \tag{14d}$$

The hybridized mixed Raviart-Thomas appoximation for the coupled problem in the form of the iterative algorithm is : get $\theta_h^0 = \theta^0$, for $n = 1, 2, \ldots$, find $(\boldsymbol{\sigma}_h^n, u_h^n, \lambda_h)$ and $(\boldsymbol{q}_h^n, \theta_h^n, \gamma_h) \in \mathcal{W}_h$, such that:

$$\left(a(\theta_h^{n-1})\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h\right)_{\mathcal{T}_h} - (u_h^n, \text{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} + \langle\lambda_h, \boldsymbol{\tau}_h.\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0,\; \forall\,\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h \tag{15a}$$

$$(\text{div}\boldsymbol{\sigma}_h^n, v_h)_{\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h},\; \forall\, v_h \in \mathcal{V}_h \tag{15b}$$

$$\langle\boldsymbol{\sigma}_h^n.\boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h} = 0,\; \forall\, m_h \in \mathcal{M}_h \tag{15c}$$

$$(\boldsymbol{q}_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} - (\theta_h^n, \text{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} + \langle\gamma_h, \boldsymbol{\tau}_h.\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0,\; \forall\,\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h \tag{16a}$$

$$(\text{div}\boldsymbol{q}_h^n, v_h)_{\mathcal{T}_h} = \left(a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, v_h\right)_{\mathcal{T}_h},\; \forall\, v_h \in \mathcal{V}_h \tag{16b}$$

$$\langle\boldsymbol{q}_h^n.\boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h} = 0,\; \forall\, m_h \in \mathcal{M}_h \tag{16c}$$

Consedering that equations (15c) and (16c) are satisfied, restrict the analysis to mixed problems:

$$\left(a(\theta_h^{n-1})\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h\right)_{\mathcal{T}_h} - (u_h^n, \text{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} = 0,\; \forall\,\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h \tag{17a}$$

$$(\text{div}\boldsymbol{\sigma}_h^n, v_h)_{\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h},\; \forall\, v_h \in \mathcal{V}_h \tag{17b}$$

$$(\boldsymbol{q}_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} - (\theta_h^n, \mathrm{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} = 0, \ \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h \tag{18a}$$

$$(\mathrm{div}\boldsymbol{q}_h^n, v_h)_{\mathcal{T}_h} = \left(a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, v_h\right)_{\mathcal{T}_h}, \ \forall v_h \in \mathcal{V}_h \tag{18b}$$

where each approximate problem satisfies the discrete versions of $Kh$-coercivity and LBB analogous to the continuous case, since for the approximation spaces defined in (14a) and (14b) it is guaranteed that $\forall v_h \in \mathcal{V}_h \, \exists \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h$ schu that $\mathrm{div}\boldsymbol{\tau}_h = v_h$ and vice versa, that is, there is a compatibility between the spaces $\boldsymbol{\Sigma}_h$ and $\mathcal{V}_h$, where

$$\boldsymbol{K_h} = \{\varsigma_h \in \boldsymbol{\mathcal{K}}_h; b(\eta_h, v_h) = (\mathrm{div}\varsigma_h, v_h)_{\mathcal{T}_h} = 0, \ \forall v_h \in \mathcal{V}_h\}, \ \text{ with } \varsigma_h \in \{\boldsymbol{\sigma}_h^n, \boldsymbol{q}_h^n\}. \tag{19}$$

where we define $\boldsymbol{\mathcal{K}_h} = \left\{\varsigma_h \in \boldsymbol{\Sigma}_h : \sum_T \int_E [\![\varsigma_h]\!] m_h = 0, \forall m_h \, \mathcal{M}_h\right\}$, $[\![\varsigma_h]\!]$ is the jump of $\varsigma_h$ in $E$. Hence $[\![\varsigma_h]\!] = 0$, therefore $\varsigma_h \in H(\mathrm{div}, \Omega)$ whith $\varsigma_h \in \{\boldsymbol{\sigma}_h^n, \boldsymbol{q}_h^n\}$. So we look for $\varsigma_h$ in $\boldsymbol{\Sigma}_h \cap H(\mathrm{div}, \Omega)$.

### 3.1.1 Error Estimates

Let's assume this convergence of iterative scheme, follow the numerical analysis based on this hypothesis. We presente below estimates for the coupled problem. For all $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h$ and $v_h \in \mathcal{V}_h$

$$\begin{aligned}&\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{H(\mathrm{div}\Omega)} + \|u^n - u_h^n\|_{(\Omega)} \leq C\left(\|\boldsymbol{\sigma}^n - \boldsymbol{\tau}_h\|_{H(\mathrm{div}(\Omega)} + \|u^n - v_h\|_{L^2(\Omega)}\right) + \\ &C|\boldsymbol{\sigma}|_\infty \|\theta^{n-1} - \theta_h^{n-1}\|_{L^2(\Omega)}\end{aligned} \tag{20}$$

where $C = C(\Omega, a_1, a_2, L)$, being $L$ Lipchitz constant.

$$\|\boldsymbol{q}^n - \boldsymbol{q}_h^n\|_{H(\mathrm{div}\Omega)} + \|\theta^n - \theta_h^n\|_{L^2(\Omega)} \leq C\left(\|\boldsymbol{q}^n - \boldsymbol{\tau}_h\|_{H(\mathrm{div}(\Omega)} + \|\theta^n - v_h\|_{L^2(\Omega)}\right) +$$

$$C\left[\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)^2\| + \|\boldsymbol{\sigma}^n\|_\infty \|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)\|\right]. \tag{21}$$

where $C = C(a_1, \Omega)$.

The exact solution $\boldsymbol{\sigma}^n$ satisfies (17a), because of iteration consistency in $n$, that is

$$\left(a(\theta^{n-1})\boldsymbol{\sigma}^n, \boldsymbol{\tau}_h\right)_{\mathcal{T}_h} - (u^n, \mathrm{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} = 0, \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h. \tag{22}$$

making the difference between (17a) and (22), we have

$$\left(a(\theta^{n-1})\boldsymbol{\sigma}^n - a(\theta_h^{n-1})\boldsymbol{\sigma}_h{}^n, \boldsymbol{\tau}_h\right)_{\mathcal{T}_h} - (u^n - u_h^n, \mathrm{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} = 0, \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h. \tag{23}$$

Adding and subtracting the term $a(\theta_h^{n-1})\boldsymbol{\sigma}^n$ in the first term of (23), we obtain

$$\begin{aligned}&\left(a(\theta_h^{n-1})(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n), \boldsymbol{\tau}_h\right)_{\mathcal{T}_h} - (u^n - u_h^n, \mathrm{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} = \left(a(\theta_h^{n-1}) - a(\theta^{n-1})\boldsymbol{\sigma}^n, \boldsymbol{\tau}_h\right), \\ &\text{for all } \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h.\end{aligned} \tag{24}$$

by the consistency of (17b) and subtracting from the approximate problem, we get,

$$(\mathrm{div}\boldsymbol{\sigma}^n - \mathrm{div}\boldsymbol{\sigma}_h^n, v_h)_{\mathcal{T}_h} = 0, \quad \forall v_h \in \mathcal{V}_h. \tag{25}$$

From (24) and (25), we get (20). The estimate (21), ís shown in more detail below. Following, Brezzi's theorem applied to this problem requires

$$(1) \quad (\boldsymbol{\tau}_h, \boldsymbol{\tau}_h)_{L^2(\Omega)} \geq \alpha\|\boldsymbol{\tau}_h\|_{H(div;\Omega)}, \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{K}_h \tag{26}$$

$$(2) \quad \sup_{\boldsymbol{\tau}_h \in \boldsymbol{K}_h} \frac{(u_h, \mathrm{div}\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{H(div;\Omega)}} \geq \beta\|u_h\|_{L^2(\Omega)}, \qquad \forall u_h \in \mathcal{V}_h \tag{27}$$

 

with $\boldsymbol{K}_h = \{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h,\, (\mathrm{div}\boldsymbol{\tau}_h, v_h)_{L^2(\Omega)} = 0,\, \forall\, v_h \in \mathcal{V}_h\}$. As consequences of space compatibility, that is, $\mathrm{div}\boldsymbol{\mathcal{K}}_h = \mathcal{V}_h$ and, therefore, for all $u_{\boldsymbol{h}}^n \in \mathcal{V}_h$, there is $\bar{\boldsymbol{\tau}}_h \in \boldsymbol{\mathcal{K}}_h$ tal que $\mathrm{div}\bar{\boldsymbol{\tau}}_{\boldsymbol{h}} = \mathcal{V}_h$, with

$$\|\bar{\boldsymbol{\tau}}_{\boldsymbol{h}}\|_{H(div;\Omega)} \leq C\|u^n\|_{L^2(\Omega)}. \tag{28}$$

It is $\tilde{\boldsymbol{q}}_h^n \in \boldsymbol{\mathcal{K}}_h$ and $\tilde{\theta}_h \in \mathcal{V}_h$ solutions of following problem

$$(\tilde{\boldsymbol{q}}_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} - (\tilde{\theta}_h, \mathrm{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} = 0,\ \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h \tag{29a}$$

$$(\mathrm{div}\tilde{\boldsymbol{q}}_h^n, v_h)_{\mathcal{T}_h} = (a(\theta^{n-1})|\boldsymbol{\sigma}^n|^2, v_h)_{\mathcal{T}_h},\ \forall\, v_h \in \mathcal{V}_h \tag{29b}$$

Subtracting (29) from (5c) and (5d) respectively, we get

$$(\boldsymbol{q}^n - \tilde{\boldsymbol{q}}_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} - (\theta - \tilde{\theta}_h, \mathrm{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} = 0,\ \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h \tag{30a}$$

$$(\mathrm{div}\boldsymbol{q}^n - \mathrm{div}\tilde{\boldsymbol{q}}_h^n, v_h)_{\mathcal{T}_h} = 0,\ \forall\, v_h \in \mathcal{V}_h. \tag{30b}$$

consequently

$$\|\boldsymbol{q}^n - \tilde{\boldsymbol{q}}_h^n\|_{H(div;\Omega)} + \|\theta^n - \tilde{\theta}_h^n\|_{L^2(\Omega)} \leq C\left(\|\boldsymbol{q}^n - \boldsymbol{q}_h^n\|_{H(div;(\Omega)} + \|\theta^n - \theta_h^n\|_{L^2(\Omega)}\right) \tag{31}$$

for all $\boldsymbol{q}_h^n \in \boldsymbol{\mathcal{K}}_h,\ \forall \in \tilde{\theta}_h^n \in \mathcal{V}_h$. From the consistency of the approximation (18), we have

$$(\boldsymbol{q}^n - \boldsymbol{q}_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} - (\theta - \theta_h, \mathrm{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} = 0,\ \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{K}}_h \tag{32a}$$

$$(\mathrm{div}\boldsymbol{q}^n - \mathrm{div}\boldsymbol{q}_h^n, v_h)_{\mathcal{T}_h} = (a(\theta^{n-1})|\boldsymbol{\sigma}^n|^2 - a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, v_h),\ \forall\, v_h \in \mathcal{V}_h. \tag{32b}$$

From (30) and (32), results

$$(\tilde{\boldsymbol{q}}^n - \boldsymbol{q}_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} - (\tilde{\theta}_h^n - \theta_h, \mathrm{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} = 0,\ \forall\, \boldsymbol{\tau}_h, \in \mathcal{K}_h \tag{33a}$$

$$(\mathrm{div}\tilde{\boldsymbol{q}}^{\boldsymbol{n}} - \mathrm{div}\boldsymbol{q}_h^n, v_h)_{\mathcal{T}_h} = (a(\theta^{n-1})|\boldsymbol{\sigma}^n|^2 - a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, v_h),\ \forall\, v_h \in \mathcal{V}_h. \tag{33b}$$

Choosing $\boldsymbol{\tau}_h = \tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n$ e $v_h = \tilde{\theta}_h^n - \theta_h$ in (33)

$$\|\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n\|_{L^2\Omega)}^2 = (a(\theta^{n-1})|\boldsymbol{\sigma}^n|^2 - a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, \tilde{\theta}_h - \theta_h^n)_{L^2(\Omega)} \tag{34}$$

choosing $\boldsymbol{\tau}_h = \bar{\boldsymbol{\tau}}_h$, such that $\mathrm{div}\bar{\boldsymbol{\tau}} = \tilde{\theta}_h^n - \theta_h$ and

$$\|\bar{\boldsymbol{\tau}}\|_{H(div;\Omega)} = C\|\tilde{\theta}_h^n - \theta_h^n\|_{L^2(\Omega)} \tag{35}$$

in (31), results

$$\|\tilde{\theta}_h^n - \theta_h^n\|_{L^2\Omega)}^2 = (\tilde{\theta}_h^n - \theta_h, \bar{\boldsymbol{\tau}}_h) \leq C\|\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n\|\|\tilde{\theta}_h^n - \theta_h^n\| \tag{36}$$

therefore

$$\|\tilde{\theta}_h^n - \theta_h^n\|_{L^2\Omega)} = (\tilde{\theta}_h^n - \theta_h, \bar{\boldsymbol{\tau}}_h) \leq C\|\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n\| \tag{37}$$

Choosing $v_h = \mathrm{div}(\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n)$ in (32), we have

$$\|\mathrm{div}(\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n)\|_{L^2(\Omega)}^2 = (a(\theta^{n-1})|\boldsymbol{\sigma}^n|^2 - a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, \mathrm{div}(\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n)) \tag{38}$$

Consedering

$$\begin{aligned} &a(\theta^{n-1})|\boldsymbol{\sigma}^n|^2 - a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2 = \\ &-(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)^2 + 2a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n) \end{aligned} \tag{39}$$

of (34), we have

 

$$\|\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n\|^2_{L^2(\Omega)} = -(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n|^2, \tilde{\theta}_h^n - \theta_h^n) + (2a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n), \tilde{\theta}_h^n - \theta_h^n) \quad (40)$$

that is

$$\|\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n\|^2_{L^2(\Omega)} \leq \{C_1\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)^2\| + C_2\|\boldsymbol{\sigma}^n\|_\infty \|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)\|\} \|\tilde{\theta}_h^n - \theta_h^n\| \quad (41)$$

combining with (35), results in

$$\|\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n\|^2_{L^2(\Omega)} \leq C\left[\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)^2\| + \|\boldsymbol{\sigma}^n\|_\infty\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)\|\right]. \quad (42)$$

From (40) and (39)

$$\|\mathrm{div}(\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n)\|_{L^2(\Omega)} \leq C\left[\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)^2\| + \|\boldsymbol{\sigma}^n\|_\infty\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)\|\right] \quad (43)$$

Combining (37), (42) and (43), we get

$$\|\tilde{\boldsymbol{q}}_h^n - \boldsymbol{q}_h^n\|_{H(div;\Omega)} + \|\tilde{\theta}_h^n - \theta_h^n\|_{L^2(\Omega)} \leq C\left[\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n\| + \|\boldsymbol{\sigma}^n\|_\infty\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)\|\right] \quad (44)$$

finally, by the triangle inequality, we get

$$\|\boldsymbol{q}^n - \boldsymbol{q}_h^n\|_{H(div;\Omega)} + \|\theta^n - \theta_h^n\|_{L^2(\Omega)} \leq C\left[\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)^2\| + \|\boldsymbol{\sigma}^n\|_\infty\|(a(\theta^{n-1})^{1/2}\boldsymbol{\sigma}^n - a(\theta_h^{n-1})^{1/2}\boldsymbol{\sigma}_h^n)\|\right] \quad (45)$$

which demonstrates the inequality (21).

## 3.2 Numerical Results

We show the numerical results obtained by solving the coupled problem (46)[3], by the proposed hybridized finite element method, in the Raviart-Thomas space $\mathcal{P}_k RTk$. Convergence studies were performed solving the problem in five triangular uniform meshs (3.2), consecutively refined, decreasing $h$. In each mesh, iterations were performed until reaching a tolerance,what is, topping criterion used tol $\leq 10^9$ , where $tol = \|u_h^{n+1} - u_h^n\|_{L^2(\Omega)} + \|\theta_h^{n+1} - \theta_h^n\|_{L^2(\Omega)}$. We calculate and analyze errors in following norms $\|u^n - u_h^n\|_{L^2(\Omega)}$, $\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{L^2(\Omega)}$ e $\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{H(div;\Omega)}$.

$$\begin{cases} \frac{1}{r}\frac{d}{dr}.(r\mu(\theta)\frac{du}{dr}) = 0 \;\; r_0 \leq r \leq r_1 \\ -\frac{1}{r}\frac{d}{dr}.(r\frac{d\theta}{dr}) = \mu(\theta)|\frac{du}{dr}|^2, \;\; r_0 \leq r \leq r_1 \\ u(r_0) = 0, \;\; u(r_1) = u_1, \\ \theta(r_0) = \theta(r_1) = 0 \end{cases} \quad (46)$$

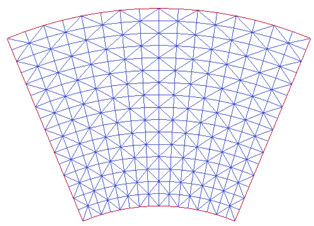

Figure 1: Discretized annular Domain

where the exact solutions are:

$$\theta(x,y) = -\ln\left[1 - \tfrac{1}{2}u(r)\,(u(r) - u_1)\right]$$
$$u(x,y) = \tfrac{1}{2}u_1 + \tfrac{1}{2}\sqrt{8+u_1^2}\tan\left[\arctan\left(\frac{u_1}{\sqrt{8+u_1^2}}\right)\left(\frac{2\ln(\frac{r}{r_0})}{\ln(\frac{r_1}{r_0})} - 1\right)\right] \quad (47)$$

with $u_1 = 1$, $r_0 = 1$ and $r_1 = 2$ e angle $\pi/4$. Tables (1) and (2) show the optimal convergence order for scalar and vector variables for each refined $h$. The errors in norms $L^2(\Omega)$ and $H(\mathrm{div}, \Omega)$ are calculated with respect to the exact solution (47).

 

Table 1: Convergence order; potential error $u$ and fluxes $\boldsymbol{\sigma}$. RTH methods of degree $k = 1$.

| $h$ | $\|u^n - u_h^n\|_{L^2(\Omega)}$ | order | $\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{L^2(\Omega)}$ | order | $\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma_h}^n\|_{H(div;\Omega)}$ | order |
|---|---|---|---|---|---|---|
| 0.7071E+00 | 0.50545E+00 | 0.00 | 0.45186E+00 | 0.00 | 0.19649E+02 | 0.00 |
| 0.3536E+00 | 0.78993E-01 | 2.68 | 0.11230E+00 | 2.01 | 0.31110E+01 | 2.66 |
| 0.1768E+00 | 0.39040E-01 | 1.02 | 0.24839E-01 | 2.18 | 0.15424E+01 | 1.01 |
| 0.8839E-01 | 0.99043E-02 | 1.98 | 0.61868E-02 | 2.01 | 0.39184E+00 | 1.98 |
| 0.4419E-01 | 0.24854E-02 | 1.99 | 0.15451E-02 | 2.00 | 0.98353E-01 | 1.99 |

Table 2: Convergence order; temperature error $\theta$ and fluxes $\boldsymbol{q}$. RTH methods of degree $k = 1$.

| $h$ | $\|\theta^n - \theta_h^n\|_{L^2(\Omega)}$ | order | $\|\boldsymbol{q}^n - \boldsymbol{q}_h^n\|_{L^2(\Omega)}$ | order | $\|\boldsymbol{q}^n - \boldsymbol{q}_h^n\|_{H(div.,\Omega)}$ | order |
|---|---|---|---|---|---|---|
| 0.7071E+00 | 0.50545E+00 | 0.00 | 0.45186E+00 | 0.00 | 0.19649E+02 | 0.00 |
| 0.3536E+00 | 0.78993E-01 | 2.68 | 0.11230E+00 | 2.01 | 0.31110E+01 | 2.66 |
| 0.1768E+00 | 0.39040E-01 | 1.02 | 0.24839E-01 | 2.18 | 0.15424E+01 | 1.01 |
| 0.8839E-01 | 0.99043E-02 | 1.98 | 0.61868E-02 | 2.01 | 0.39184E+00 | 1.98 |
| 0.4419E-01 | 0.24854E-02 | 1.99 | 0.15451E-02 | 2.00 | 0.98353E-01 | 1.99 |

## 4 Final Considerations

The hypothesis made in the hybridization of the discrete form provided the numerical analysis of the mixed form concise and intelligible, arriving at the error estimates without extensive mathematical complications.

Computational experiments, using the RTH method with degree k=1, confirm the rates of optimal convergence $k+1$ for the flows in $H(\text{div};\Omega)$ and for temperature in $L^2(\Omega)$.

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