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Quotient space of intervals

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Abstract. This article studies the quotient space of intervals, for this is defined an equivalence relation considering a symmetric difference, is obtained the quotient space of intervals where is defined a specific representative of the equivalence class, and an appropriated norm and metric is defined to proof that this space is a complete metric space.

Keywords. Interval space, linear space of intervals, quotient space of intervals.

1 Introduction

From interval analysis it is well known that the space of intervals provided with the standard sum and the product of a scalar forms a quasi-vector space [1, 7]. This is because, for example, an interval does not have inverse element and therefore subtraction does not have many useful properties (see [2, 8]). But if we define a specific equivalence relation we can establish an equivalence class in such a way that it is possible to construct a quotient space of intervals [5, 9], in order to develop an interval mathematical analysis.

Radström's embedding theorem [6] tells us that there is an isometric mapping $\pi : \mathbb{I} \to \mathcal{B}$, where $\mathcal{B}$ is a real normed linear space (space of equivalence classes), and $\mathbb{I}$ is the family of all bounded closed intervals [6]. Taking a norm to induce a metric we can prove that this is a complete metric space.

The paper is organized as follows. Section 2 introduces the an equivalence relation to determine the equivalence class, then a specific representative is defined and then is defined the quotient space, and in Section 3 we present the conclusion.

2 The quotient space of intervals

Let $\mathbb{I}$ is the family of all bounded closed intervals, on this space it is well known that the addition is associative, commutative and its neutral element is $\{0\}$. For $\lambda = -1$, scalar multiplication gives the opposite $-A = (-1)A = \{-a : a \in A\}$ but, in general, $A + (-)A \neq \{0\}$, that is, the space $\mathbb{I}$ is not a linear space.

This fact is a crucial point due the necessity of working on a linear space in order to define in a suitable sense the derivative of interval valued functions. Taking into account this problem, we will introduce a natural equivalence relation between elements of $\mathbb{I}$ which can be used to divide $\mathbb{I}$ into equivalence classes having group properties for the addition operation. Building a vector space over the elements of $\mathbb{I}$ is important, as this will allow us to define important limits such as

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derivative and integral, in such a way that they have important properties such as the linearity of the derivative, a property that we do not have when we talk about gH-derivative for example [3].

Given $A = [\underline{a}, \overline{a}] \in \mathbb{I}$, the interval A is called symmetric if $\underline{a} = -\overline{a}$ and the class of symmetric intervals of $\mathbb{I}$ will be denoted by $\mathcal{S}$. Given $A, B \in \mathbb{I}$, where $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$, the standard difference is defined by $A - B = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$.

On next, we define a convenient relation such as the quotient space obtained from this relation will be isomorphic to $\mathbb{R}$.

Definition 2.1. *Let $A, B \in \mathbb{I}$, $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$. $\sim$ is a relation on $\mathbb{I}$ and A is in relation with B, and write $A \sim B$, if $A - B \in \mathcal{S}$.*

Remark 2.1. *Let $A, B \in \mathbb{I}$, $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$. If $[\underline{a}, \overline{a}] - [\underline{b}, \overline{b}] = [-c, c] \in \mathcal{S}$ then $\underline{a} - \overline{b} = -c$ and $\overline{a} - \underline{b} = c$; that is, $\underline{a} + \overline{a} = \underline{b} + \overline{b}$. Thus, considering $M_A = \underline{a} + \overline{a}, M_B = \underline{b} + \overline{b}$, we obtain that, $A \sim B$ if and only if $M_A = M_B$.*

Note that, given $A, B, C \in \mathbb{I}$, $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$ and $C = [\underline{c}, \overline{c}]$, then

- $(A, A) \in \sim$ because $M_A = M_A$, then $\sim$ is reflexive.
- $A \sim B$ implies that $M_A = M_B$ then $B \sim A$, i.e. $\sim$ is symmetric.
- $A \sim B$ and $B \sim C$ implies that $M_A = M_B = M_C$ then $A \sim C$.

Therefore, the relation $\sim$ is an equivalence relation, that is, $\sim$ is reflexive, symmetric and transitive. We will denote by $\langle A \rangle$ the equivalence class containing the interval $A \in \mathbb{I}$. The set of equivalence classes will be denoted by $\mathbb{I}/\mathcal{S}$. Note that if $\langle A \rangle \in \mathbb{I}/\mathcal{S}$, and considering $A = [\underline{a}, \overline{a}] = [a]$ when $\underline{a} = \overline{a}$ it is obtained the degenerated intervals, therefore, we can choose the representative of the class as

$$\begin{aligned} \langle A \rangle &= \langle [\underline{a}, \overline{a}] \rangle \\ &= \left\langle \left[\frac{\underline{a} + \overline{a}}{2} \right] \right\rangle. \end{aligned} \tag{1}$$

In particular, $\mathcal{S} = \langle [0, 0] \rangle := \langle 0 \rangle$ represents the class of all symmetric intervals.

Graphically, if we consider $A \in \mathbb{I}$, considering the extremes of this interval as ordered pairs, we have that they can be represented by points in the half plane above the $x = y$ line. For our study in question, for example, if we consider $A = [-1, 3]$, we will have that the class of A is represented by a degenerate interval, that is, $\langle [-1, 3] \rangle = \left\langle \left[\frac{-1 + 3}{2} \right] \right\rangle = \langle [1] \rangle$, thus, geometrically $\langle [1] \rangle$ represents all intervals whose endpoints are on the ray perpendicular to the identity at the point $(1, 1) \in \mathbb{R}^2$, see Figure 1.

For any $\langle A \rangle, \langle B \rangle \in \mathbb{I}/\mathcal{S}$ we define the addition $\langle A \rangle + \langle B \rangle$ by

$$\langle A \rangle + \langle B \rangle = \langle A + B \rangle.$$

Therefore $M_{A+B} = M_A + M_B$.

Lemma 2.1. *$(\mathbb{I}/\mathcal{S}, +)$ is a group, that is, for any $\langle A \rangle, \langle B \rangle, \langle C \rangle \in \mathbb{I}/\mathcal{S}$,*

(i) $\langle A \rangle + \langle B \rangle = \langle B \rangle + \langle A \rangle$

(ii) $(\langle A \rangle + \langle B \rangle) + \langle C \rangle = \langle A \rangle + (\langle B \rangle + \langle C \rangle)$

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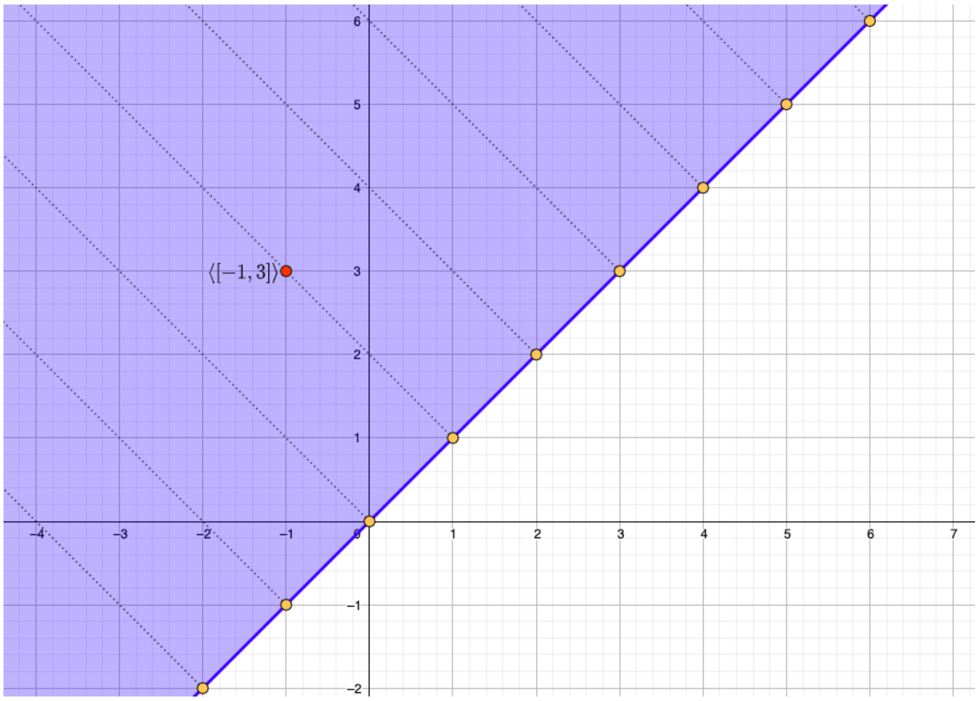

Figure 1: Quotient Space of Intervals

(iii) $\langle A\rangle + \langle B\rangle = \langle A\rangle$ *if and only if* $\langle B\rangle = \langle 0\rangle$

(iv) $\langle A\rangle + \langle B\rangle = \langle 0\rangle$ *if and only if* $\langle A\rangle = \langle -B\rangle$.

We want to remark that $\{C = A + B : A \in \langle A\rangle,\ B \in \langle B\rangle\} = \langle A\rangle + \langle B\rangle$. Multiplication of an element of $\mathbb{I}/\mathcal{S}$ by a real number λ is the following:

$$\lambda \cdot \langle A\rangle = \langle \lambda \cdot A\rangle.$$

From Lemma 2.1, for any $A \in \mathbb{I}$, we have that $-\langle A\rangle = \langle -A\rangle$, is the additive inverse of $\langle A\rangle$. In particular, $1\langle A\rangle = \langle A\rangle$. Moreover, $\lambda M_A = M_{\lambda A}$ and the following properties hold.

Lemma 2.2. *For any* $\langle A\rangle, \langle B\rangle \in \mathbb{I}/\mathcal{S}$ *and* $c_1, c_2 \in \mathbb{R}$, *the following statements hold:*

(i) $(c_1 c_2) \cdot \langle A\rangle = c_1 \cdot (c_2 \cdot \langle A\rangle)$

(ii) $c_1 \cdot (\langle A\rangle + \langle B\rangle) = c_1 \cdot \langle A\rangle + c_1 \cdot \langle B\rangle$

(iii) $(c_1 + c_2) \cdot \langle A\rangle = c_1 \cdot \langle A\rangle + c_2\langle A\rangle$.

Proof The proof follows immediately from the Remark 2.1. □

From Lemma 2.1 and Lemma 2.2 we obtain the following lemma.

Theorem 2.1. $(\mathbb{I}/\mathcal{S}, +, \cdot)$ *is a linear space.*

Now we will give other properties on the interval operations. First of all, we recall that if $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}] \in \mathbb{I}$, the gH-difference $A \ominus_{gH} B$ is defined as follows

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a)\ A = B + C \text{ or} \\ (b)\ B = A + (-1)C. \end{cases} \tag{2}$$

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In case (a), the gH-difference coincides with the well-known H-difference. Moreover, the gH-difference exists for any two compact intervals. [4] showed that the gH-difference and the π-difference between the intervals $A, B \in \mathbb{I}$ are the same concept. Specifically,

$$A \ominus_{gH} B = A -_{\pi} B = \left[\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}\right]. \quad (3)$$

Proposition 2.1. *Let $\langle A \rangle, \langle B \rangle \in \mathbb{I}/\mathcal{S}$. Then*

$$\langle A \ominus_{gH} B \rangle = \langle A \rangle - \langle B \rangle.$$

Proof. Given $A, B \in \mathbb{I}$, taking into account (2) and (1), we have that, the class $\langle A \ominus_{gH} B \rangle$ is characterized by $M_{A \ominus_{gH} B} = \dfrac{(\underline{a} - \underline{b}) + (\overline{a} - \overline{b})}{2} = \dfrac{(\underline{a} + \overline{a}) - (\underline{b} + \overline{b})}{2} = \dfrac{\underline{a} + \overline{a}}{2} - \dfrac{\underline{b} + \overline{b}}{2} = M_A - M_B$ then $\langle A \ominus_{gH} B \rangle = \langle A \rangle - \langle B \rangle$. □

We now provide a norm $\| \cdot \|$ on the space $\mathbb{I}/\mathcal{S}$.

Definition 2.2. *Let $\langle A \rangle = \langle [\underline{a}, \overline{a}] \rangle \in \mathbb{I}/\mathcal{S}$. We define the norm of $\langle A \rangle$ by*

$$\|\langle A \rangle\| = |\underline{a} + \overline{a}|.$$

Remark 2.2. *$(\mathbb{I}/\mathcal{S}, \| \cdot \|)$ is a normed linear space. Moreover, we have the metric d_{sup} on $\mathbb{I}/\mathcal{S}$ defined by*

$$d_{sup}(\langle A \rangle, \langle B \rangle) = \|\langle A \rangle - \langle B \rangle\|,$$

for all $\langle A \rangle, \langle B \rangle \in \mathbb{I}/\mathcal{S}$. Notice that for $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$, $d_{sup}(\langle A \rangle, \langle B \rangle) = |(\underline{a} + \overline{a}) - (\underline{b} + \overline{b})|$.

The following properties is a immediate consequence.

Proposition 2.2. *Let $\langle A \rangle, \langle B \rangle, \langle C \rangle \in \mathbb{I}/\mathcal{S}$. Then d_{sup} is translation invariant, that is,*

$$d_{sup}(\langle A \rangle + \langle C \rangle, \langle B \rangle + \langle C \rangle) = d_{sup}(\langle A \rangle, \langle B \rangle);$$

Lemma 2.3. *$(\mathbb{I}/\mathcal{S}, d_{sup})$ is a complete metric space.*

3 Conclusion

This article introduced a quotient space of closed and bounded intervals with respect to the family of symmetric intervals. We showed that it is a normed linear space. Since the space of closed and bounded intervals can be embed on this quotient space, we introduced a concept of metric to proof that these space is a complete metric space.

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