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Unidirectional Flows of Ferrofluids in the Presence of Magnetic Fields

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It is widely known that the Navier-Stokes equations encompass the mass and momentum principles of Physics when modeling the flow of viscous Newtonian fluids. Assuming no gravitational effects on *steady-state incompressible flows*, those equations can be written as

$$-\nabla p + \eta \nabla^2 \mathbf{u} + \rho \mathbf{F} = 0, \tag{1}$$

where ρ and η are the fluid's density and dynamic viscosity, both assumed to be constant, p is the pressure exerted on it, and vectors $\mathbf{u}$ and $\mathbf{F}$ represent the velocity and total externally applied conservative force, respectively, as the fluid flows through the infinite channel depicted in figure 1.

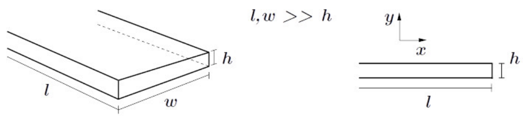

Figure 1: Geometry of the channel of dimensions $l \times w \times h$ where fluid of interest flows through.

Here an electrically nonconducting magnetic polarized fluid is investigated when submitted to a force $\mathbf{F}$ originated from an applied magnetic field $\mathbf{H}$, thereafter named *ferrofluid* [2]. Its response to $\mathbf{H}$ introduces extra terms in (1) namely, from left to right, the *Kelvin Force* and *torque term* [1]:

$$0 = -\nabla \rho + \nu \nabla^2 \mathbf{u} + \mu_0 \mathbf{M} \cdot \nabla \mathbf{H} \; + \frac{1}{2}\mu_0 \nabla \times (\mathbf{M} \times \mathbf{H}), \tag{2}$$

where the magnetic permeability of vacuum is μ_0. Given its complexity, exact analytical solutions of (2) are not available and certain simplifying hypotheses become mandatory in order to understand the underlying dynamics. Thus assuming fluid velocity as bidirectional of the type $\mathbf{u} = (u(x, y), v(x, y), 0)$ and pressure p of the form $p = p(x, y, 0)$ and considering, for a moment, complete absence of any magnetic field, we find out that $v \ll u$, provided the narrow geometry of the channel since $h/l \ll 1$. Additionally, resorting to scale analysis (a very usual tool in fluid dynamics), we can ascertain that $\partial^2 u/\partial x^2 \ll \partial^2 u/\partial y^2$ and $\partial^2 v/\partial x^2 \ll \partial^2 v/\partial y^2$, so horizontal and vertical components of (2), respectively, result in $-\partial p/\partial x + \nu \partial^2/\partial y^2 = 0$ and $\partial p/\partial y \approx 0$, and a special *unidirectional flow hypothesis* is admitted with $\mathbf{u} = (u(x, y), 0, 0)$ and $p = p(x, 0, 0)$. In the case of a *fully developed flow* (when u is a function of y only) and non-slip boundary conditions are met, i.e., $u(x = 0, y = 0) = u(x = 0, y = h) = 0$, we finally obtain $d^2u/dy^2 = G/\nu$ (where G is the constant pressure gradient), which has the simple exact solution $u(y) = (G/2\nu)(y - h)y$, a parabolic velocity profile well known as *Poiseuille flow.*

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Turning back to the more interesting problem where $\mathbf{H}$ is indeed present, Maxwell's equations must also be satisfied at the walls between the ferrofluid and surrounding media under null boundary conditions for the transversal component of the magnetic flux density $\mathbf{B}$ and the tangential component of $\mathbf{H}$, where $\mathbf{B} = \mu_0(\mathbf{M} + \mathbf{H})$ and $\mathbf{M}$ is the *magnetization vector* comprising the ferrofluid's full response to $\mathbf{H}$. Shliomis [3] proposed an evolving equation for $\mathbf{M}$ given by

$$\frac{\partial \mathbf{M}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{M} = -\frac{1}{\tau}(\mathbf{M} - \mathbf{M}_{eq}) + \frac{1}{2}(\nabla \times \mathbf{u}) \times \mathbf{M} + \frac{1}{6\phi}\frac{\mu_0}{\nu}(\mathbf{M} \times \mathbf{H}) \times \mathbf{M}, \tag{3}$$

where ϕ is the particle volume fraction, τ is the relaxation time and $\mathbf{M}_{eq} = M_S \mathcal{L}(\alpha)\mathbf{H}/|\mathbf{H}|$ is the *equilibrium magnetization vector* which, in turn, depends on the saturation magnetization M_S and the *Langevin function* $\mathcal{L}(\alpha) = cotgh(\alpha) - 1/\alpha$, with $\alpha = m|\mathbf{H}|/k_B T$, with m being the magnetic momentum of the fluid's particles, T its temperature and k_B the Boltzmann constant.

It can be shown that using the aforementioned conditions impose several partial derivatives in (2) and (3) to vanish and therefore, after applying the earlier unidirectional flow hypothesis, magnetic field and magnetization vectors take the simplified form of $\mathbf{H} = (H_x, H_y(y), 0)$ and $\mathbf{M} = \big(M_x(y), M_y(y), 0\big)$, respectively, where subscripts emphasize the component directions.

It is desirable to make the resulting Navier-Stokes equations dimensionless by choosing proper scales for selected variables. As such, we state the following discretionary set of scales $\{y \sim h, u \sim U, p \sim P, H_x, H_y, M_x, M_y \sim H_0\}$, where $H_0 = (H_x^2 + H_y^2|_{y=0})^{1/2}$ is the necessary constraint that normalizes the peak value of $\mathbf{H}$, P is the pressure scale drawn from Newton's law of viscosity $\tau = \eta\, du/dy$, and U is the velocity scale chosen from the Poiseuille flow for the non-magnetic case. The derived dimensionless parameteres are the *Péclet number* $P_e = \tau/(h/U)$, $\alpha_0 = \alpha/|\mathbf{H}|$, and $G_\eta = -Gh^2/\eta = 8$, assuring unitary maximum velocity halfway through the channel height h).

Dimensionless Navier-Stokes equations can then be presented as a set of two nonlinear algebraic equations on the variables M_x and M_y and a second-order linear Boundary Value Problem on u as

$$\begin{aligned}
M_x: &\quad \frac{1}{2}M_y\frac{du}{dy} + \frac{1}{2\phi}\left(\frac{\alpha_0}{P_e\chi_M}\right)\Big[M_xH_y - M_yH_x\Big]M_y - \frac{1}{P_e}\Big[M_x - M_{eq_x}\Big] = 0\\
M_y: &\quad -\frac{1}{2}M_x\frac{du}{dy} - \frac{1}{2\phi}\left(\frac{\alpha_0}{P_e\chi_M}\right)\Big[M_xH_y - M_yH_x\Big]M_x - \frac{1}{P_e}\Big[M_y - M_{eq_y}\Big] = 0\\
u: &\quad \frac{d^2u}{dy^2} + \frac{3}{2}\left(\frac{\alpha_0}{P_e\chi_M}\right)\frac{d}{dy}\Big[M_xH_y - M_yH_x\Big] - G_\eta = 0, \quad u(0) = u(1) = 0,
\end{aligned} \tag{4}$$

over the intervals $0 \le y \le 1$, $0 \le H_x, H_y(y) \le 1$, and $0 \le M_x, M_y \le \phi\chi_M$, where $\chi_M = M_S/H_0$ is the magnetic susceptibility, $M_{eq_x} = \phi\chi_M\mathcal{L}(\alpha_0 H)H_x/|\mathbf{H}|$, and $M_{eq_y} = \phi\chi_M\mathcal{L}(\alpha_0 H)H_y/|\mathbf{H}|$.

We aim at numerically solving our final model (4) by using a classic *Newton-Raphson* method to calculate the local values of M_x and M_y from the algebraic 2×2 nonlinear system present in the x- and y- directions above, and fulfilling the boundary conditions in the x-direction to solve the differential equation using the *shooting method*.

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