Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 3, N. 1, 2015.
Trabalho apresentado no XXXV CNMAC, Natal-RN, 2014.

On the Poincaré map and limit cycles for a class of continuous piecewise linear differential systems with three zones

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Resumo: *In this work, we are interested in study the existence of limit cycles for the class of continuous piecewise linear differential systems with three zones*

$$\mathbf{x}' = X(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, *and* X *is a continuous piecewise linear vector field. For this, we make a thorough analysis of the Poincaré map for such vector fields.*

Palavras-chave: *piecewise vector field, limit cycles, Poincaré map.*

1 Introduction

Due to the encouraging increase in their applications, control theory [5] and [9], design of electric circuits [2], neurobiology [4] and [8] piecewise linear differential systems were studied early from the point of view of qualitative theory of ordinary differential equations [1]. Nowadays, a lot of papers are being devoted to these differential systems.

On the other hand, starting from linear theory, in order to capture nonlinear phenomena, a natural step is to consider piecewise linear systems. As local linearizations are widely used to study local behavior, global linearizations (achieved quite naturally by working with models which are piecewise linear) can help to understand the richness of complex phenomena observed in the nonlinear world.

The study of piecewise linear systems can be a difficult task that is not within the scope of traditional nonlinear systems analysis techniques. In particular, a sound bifurcation theory is lacking for such systems due to their nonsmooth character.

In this work, we study the existence of limit cycles for the class of continuous piecewise linear differential systems

$$\mathbf{x}' = X(\mathbf{x}), \tag{2}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and X is a continuous piecewise linear vector field.

We will consider the following situation, that we will name the three-zone case. We have two parallel straight lines L_- and L_+ symmetric with respect to the origin dividing the phase plane in three closed regions: R_-, R_o and R_+ with $(0,0) \in R_o$ and the regions R_- and R_+ have as boundary the straight lines L_- and L_+ respectively. We will denote by X_- the vector field X restrict to R_-, by X_o the vector field X restricted to R_o and by X_+ the vector field X restrict to R_+. We suppose that the restriction of the vector field to each one of these zones are linear systems with constant coefficients that are glued continuously at the common boundary.

In short, system (2) can be written as

$$\mathbf{x}' = \begin{cases} A_-\mathbf{x} + B_- & \mathbf{x} \in R_-, \\ A_o\mathbf{x} + B_o & \mathbf{x} \in R_o, \\ A_+\mathbf{x} + B_+ & \mathbf{x} \in R_+, \end{cases} \tag{3}$$

where $A_i \in \mathcal{M}_2(\mathbb{R}),\ i \in \{-, o, +\}$, $B_i \in \mathbb{R}^2,\ i \in \{-, o, +\}$ and $\mathbf{x}' = \dfrac{d\mathbf{x}}{dt}$ with t the time.

We say that the vector field X_i has a *real equilibrium* x^* in R_i with $i \in \{-, o, +\}$ if x^* is an equilibrium of X_i and $x^* \in R_i$. In opposite we will say that X_i has a *virtual equilibrium* x^* in R_i if x^* is an equilibrium of X_i and $x^* \in R_i^c$, where R_i^c denotes the complementary of R_i in $\mathbb{R}^2$.

We suppose the following assumptions:

(H1) X_o has an equilibrium of focus type.

(H2) The others equilibria, of X_- and X_+, are a center and a focus with different stability with respect to the focus of X_o.

We note that if the two equilibria of X_- and X_+ are both centers, or a center and a focus having this focus the same stability than the focus of X_o, then the vector field X has no limit cycles, for more details see Proposition 1.

We denote t_i the trace of matrix A_i, and by d_i the determinant of the matrix A_i, for $i \in \{-, o, +\}$.

Proposition 1. *If system* (4) *has a simple invariant closed curve* Γ *then*

$$\iint_{Int_-(\Gamma)} t_- dxdy + \iint_{Int_o(\Gamma)} t_o dxdy +$$

$$\iint_{Int_+(\Gamma)} t_+ dxdy = t_- S_- + t_o S_o + t_+ S_+ = 0,$$

where $Int(\Gamma)$ *is the open region limited by the closed Jordan curve* Γ, $Int_i(\Gamma) = Int(\Gamma) \cap R_i$ *and* $S_i = area(Int_i(\Gamma))$ *with* $i \in \{-, o, +\}$.

As usual a *limit cycle* of (3) is a periodic orbit of (3) isolated in the set of all periodic orbits of (3). A limit cycle is *hyperbolic* if the integral of the divergent of the system along it is different from zero, for more details see for instance [3].

2 Normal Form

The next result give us system (3) in a convenient normal form where the number of parameters are reduced, and consequently the computations of the Poincaré return map will be easier.

Lemma 2. *Suppose that system* (3) *is such that* $det(A_o) > 0$. *Then there exists a linear change of coordinates that writes system* (3) *into the form*

$$\dot{\mathbf{x}} = X(\mathbf{x}),$$

with $L_- = L_{-1} = \{x = -1\}$, $L_+ = L_1 = \{x = 1\}$, $R_- = \{(x, y) \in \mathbb{R}^2;\ x \leq -1\}$, $R_o = \{(x, y) \in \mathbb{R}^2;\ -1 \leq x \leq 1\}$, $R_+ = \{(x, y) \in \mathbb{R}^2;\ x \geq 1\}$ *and*

$$X(\mathbf{x}) = \begin{cases} A_-\mathbf{x} + B_- & \mathbf{x} \in R_-, \\ A_o\mathbf{x} + B_o & \mathbf{x} \in R_o, \\ A_+\mathbf{x} + B_+ & \mathbf{x} \in R_+, \end{cases} \tag{4}$$

DOI: 10.5540/03.2015.003.01.0038

where $A_- = \begin{pmatrix} a_{11} & -1 \\ 1-b_2+d_2 & a_1 \end{pmatrix}$, $B_- = \begin{pmatrix} a_{11} \\ d_2 \end{pmatrix}$, $A_o = \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix}$, $B_o = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$, $A_+ = \begin{pmatrix} c_{11} & -1 \\ 1+b_2-f_2 & a_1 \end{pmatrix}$ *and* $B_+ = \begin{pmatrix} -c_{11} \\ f_2 \end{pmatrix}$. *The dot denotes derivative with respect to a new time* s.

Proof. By means of a rotation and a homothecy in the x direction we can write the system in such a way that $L_- = L_{-1}$, $L_+ = L_1$. Now doing the change of coordinates given by $u = x$, $v = k_1x + k_2y + k_3$ and the time rescaling $t = k_4s$ with

$$k_1 = -\frac{b_{11}}{\sqrt{b_{11}b_{22} - b_{12}b_{21}}},\ k_2 = -\frac{b_{12}}{\sqrt{b_{11}b_{22} - b_{12}b_{21}}},$$

$$k_3 = -\frac{e_1}{\sqrt{b_{11}b_{22} - b_{12}b_{21}}},\ k_4 = \frac{1}{\sqrt{b_{11}b_{22} - b_{12}b_{21}}},$$

where b_{ij}, $i, j = 1, 2$ are the initial entries of the matrix A_o and $B_o = (e_1, e_2)$, we have (4). □

3 Poincaré map

For our purpose we will define a first return map that involves all the vector fields X_-, X_o and X_+ in a suitable transversal section.

In [6] and [7], a necessary and sufficient condition for the existence of Poincaré maps from the straight lines $L_\pm$ to the straight lines $L_\pm$ is that there exists a unique contact point of the flow of the linear system with these lines. By *contact point* we mean a point of the line where the vector field is tangent to it.

Lemma 3. *In the coordinates given by Lemma 2 there is a unique contact point of system* (4) *with* L_- *and a unique contact point of* (4) *with* L_+. *These points are respectively* $p_- = (-1, 0)$ *and* $p_+ = (1, 0)$. *Moreover under the assumptions (H1) and (H2), we have*

(i) if $b_2 < -1$, *the equilibriums points of* X_- *and* X_0 *are virtual and the equilibrium point of* X_+ *is real;*

(ii) if $b_2 = -1$, X_0 *and* X_+ *have an equilibrium point at* $(1, 0)$, *and* X_- *has a virtual equilibrium point;*

(iii) if $|b_2| < 1$, *the equilibriums points of* X_- *and* X_+ *are virtual and the equilibrium point of* X_0 *is real;*

(iv) if $b_2 = 1$, X_- *and* X_0 *have an equilibrium point at* $(-1, 0)$, *and* X_+ *has a virtual equilibrium point;*

(v) if $b_2 > 1$, *the equilibriums points of* X_0 *and* X_+ *are virtual and the equilibrium point of* X_- *is real.*

In the rest, we are considering $b_2 < -1$.

The Poincaré return map will be defined as the composition of four different Poincaré maps. In what follows we study the qualitative behavior of each one of these maps separately in order to understand the global behavior of the general Poincaré return map.

Let p_- be the contact point of $\dot{\mathbf{x}} = A_-\mathbf{x} + B_-$ with L_-. Note that p_- divides L_- into two segments L_-^O and L_-^I where in L_-^O the vector field points toward the region R_- while in L_-^I the vector field points toward the region R_o. In fact we have $L_-^O = \{(-1, y);\ y \geq 0\}$ and $L_-^I = \{(-1, y);\ y \leq 0\}$.

We can define a Poincaré map $\Pi_- : L_-^O \to L_-^I$ by $\Pi_-(p) = q$ as the first return map in forward time of the flow of $\dot{\mathbf{x}} = A_-\mathbf{x} + B_-$ to L_-, that is, if $\varphi_-(s, p)$ is the solution of $\dot{\mathbf{x}} = A_-\mathbf{x} + B_-$

such that $\varphi_-(0,p) = p$ and $p \in L_-^O$, then $q = \varphi_-(s,p)$, $s \geq 0$ such that $q \in L_-^I$. Observe that $\Pi_-(p_-) = p_-$.

We can see the mapping Π_- in a different way as follows. Given, $p \in L_-^O$ and $q \in L_-^I$ there exist unique $a \geq 0$ and $b \geq 0$ such that $p = p_- - a\dot{p}_-$ where $\dot{p}_- = X_-(p_-) = (0, b_2 - 1)$, and $q = p_- + b\dot{p}_-$. So the mapping Π_- induces a mapping π_- given by $\pi_-(a) = b$.

Note that to study the qualitative behavior of Π_- is equivalent to study the qualitative behavior of π_-. From now on we will consider the map π_- instead of Π_-.

Note that we can define in the same way a map π_+ associated to the a Poincaré map Π_+ in L_+ considering the flow defined by $\dot{\mathbf{x}} = A_+\mathbf{x} + B_+$ and the contact point p_+.

3.1 Different types of Poincaré maps

In what follows, we present the study of Poincaré return maps defined in each zone. We denote the matrix A_+ by $A_+ = \begin{pmatrix} \alpha_+ & -\beta_+ \\ \beta_+ & \alpha_+ \end{pmatrix}$, $\tau_+ = \beta_+ s$ and $\gamma_+ = \alpha_+/\beta_+$.

Proposition 4. *Consider the vector field X_- in R_- with a virtual center. Let π_- be the map associated to the Poincaré map $\Pi_- : L_- \to L_-$ defined by the flow of the linear system $\dot{\mathbf{x}} = A_-\mathbf{x} + B_-$, then $\pi_-(a) = -a$.*

Proposition 5. *Consider the vector field X_+ in R_+ with a real focus equilibrium and such that $t_+ > 0$. Let π_+ be the map associated to the Poincaré map $\Pi_+ : L_+ \to L_+$ defined by the flow of the linear system $\dot{\mathbf{x}} = A_+\mathbf{x} + B_+$.*

(a) If $t_+ > 0$ then the maps π_+ satisfy that $\pi_+ : [0,\infty) \to [b^*,\infty)$, $\pi_+(0) = b^* > 0$, $\lim_{a\to\infty} \pi_+(a) = +\infty$ and $\pi_+(a) > a$ in $(0,\infty)$.

(a.1) If $a \in (0,\infty)$ then $(\pi_+)'(a) = \dfrac{a}{\pi_+(a)} e^{2\gamma_+\tau_+}$. Moreover $(\pi_+)'(a) > 0$ and $\lim_{a\to 0}(\pi_+)'(a) = 0$.

(a.2) If $a \in (0,\infty)$ then $(\pi_+)''(a) > 0$.

(a.3) The straight line $b = e^{\gamma_+\pi}a + t_+(1+e^{\gamma_+\pi})/d_+$ in the plane (a,b) is an asymptote of the graph of π_+ when a tends to $+\infty$ where $\gamma_+ = t_+/\sqrt{4d_+ - t_+^2}$. So $\lim_{a\to\infty}(\pi_+)'(a) = e^{\gamma_+\pi}$.

Proof. Let p_+ be the contact point of the flow with L_+ and p and q as described above. As q is in the orbit of p in the forward time we have that $q = \varphi(s,p)$ with $s \geq 0$. Moreover for computing the map π_+ we can suppose that the real equilibrium is at the origin and that matrix A_+ is in its real Jordan normal form.

Let p_+^* be the contact point p_+ in the coordinates in which A_+ is in its real Jordan normal form and the virtual equilibrium of X_+ is at the origin. We denote by $\dot{p}_+^* = X_+(p_+^*)$. So we can write

$$q = \varphi_+(s,p) = e^{A_+ s}p.$$

As $p = p_+^* + a\dot{p}_+^*$ and $q = p_+^* - \pi_+(a)\dot{p}_+^*$ we obtain

$$p_+^* - \pi_+(a)\dot{p}_+^* = e^{A_+ s}(p_+^* + a\dot{p}_+^*).$$

Now using the fact that $\dot{p}_+^* = A_+p_+^*$ we have

$$(Id - \pi_+(a)A_+)p_+^* = e^{A_+ s}(Id + aA_+)p_+^*, \tag{5}$$

where $a \geq 0$, $\pi_+(a) \geq 0$, $s \geq 0$ and the matrix A_+ is given by $A_+ = \begin{pmatrix} \alpha_+ & -\beta_+ \\ \beta_+ & \alpha_+ \end{pmatrix}$ with $\alpha_+ = \dfrac{t_+}{2}$. Since $p_+^* \neq (0,0)$ we obtain from equation (5) that $b = \pi_+(a)$ is defined by the system

$$\begin{aligned} 1 - b\alpha_+ &= e^{\alpha_+ s}(\cos(\beta_+ s) + a[\alpha_+\cos(\beta_+ s) - \beta_+\sin(\beta_+ s)]), \\ b\beta_+ &= -e^{\alpha_+ s}(\sin(\beta_+ s) + a[\alpha_+\sin(\beta_+ s) + \beta_+\cos(\beta_+ s)]), \end{aligned} \tag{6}$$

and the inequalities $a \geq 0$, $b \geq 0$ and $s \geq 0$.

Let be $\pi_+(0) = b^*$, we have that $b^* > 0$. Moreover if $a = a_o$, $b = b_o$ and $s = s_o$ is a solution of (6), then s_o is the flight time between the points $p = p_+^* + a\dot{p}_+^*$ and $q = p_+^* - b\dot{p}_+^*$.

Define $\tau_+ = \beta_+ s$ and $\gamma_+ = \alpha_+/\beta_+$. Solving system (6) with respect to τ_+ we obtain the following parametric equations for $\pi_+(a) = b$,

$$a(\tau_+) = -\frac{\beta_+ e^{-\gamma_+\tau_+}}{d_+ \sin\tau_+}\varphi(\tau_+, \gamma_+) \quad \text{and}$$
$$b(\tau_+) = -\frac{\beta_+ e^{\gamma_+\tau_+}}{d_+ \sin\tau_+}\varphi(\tau_+, -\gamma_+), \tag{7}$$

where φ is the function given by $\varphi(x, y) = 1 - e^{xy}(\cos x - y\sin x)$. Since A is given in its real Jordan normal form, τ_+ is the angle covered by the solution during the flight time s. Hence we conclude that $\tau_+ \in (\pi, \tau^*)$, where $\tau^* < 2\pi$. Note that τ^* is the angle covered by the solution during the flight time s^*, i.e. $e^{s^*A}p_+^* = \Pi_+(p_+^*)$.

Now since $\lim\limits_{\tau_+\to\pi^+} a(\tau_+) = +\infty$ and $\lim\limits_{\tau_+\to\pi^+} b(\tau_+) = +\infty$ it follows that the domain of definition of π_+ is $[0, +\infty)$ and $\lim\limits_{a\to\infty} \pi_+(a) = +\infty$.

Moreover when $\tau_+ \in (\pi, \tau^*)$ we have

$$b(\tau_+) - a(\tau_+) = -\frac{2\beta_+}{d_+ \sin\tau_+}(\sinh(\gamma_+\tau_+) - \gamma_+ \sin\tau_+).$$

Since $\sinh(\gamma_+\tau_+) > \gamma_+ \sin\tau_+$ when $\tau_+ \in (\pi, \tau^*)$, we conclude from the expression above that $b(\tau_+) > a(\tau_+)$ if $\tau_+ \in (\pi, \tau^*)$. Therefore $\pi_+(a) > a$ in $(0, +\infty)$. So statement (a) is proved.

Derivating (7) with respect to τ_+ it follows that

$$\frac{da}{d\tau_+} = -\frac{\beta_+}{d_+ \sin^2\tau_+}\varphi(\tau_+, -\gamma_+) \quad \text{and}$$
$$\frac{db}{d\tau_+} = -\frac{\beta_+}{d_+ \sin^2\tau_+}\varphi(\tau_+, \gamma_+).$$

Thus $(\pi_+)'(a) = \dfrac{\varphi(\tau_+, \gamma_+)}{\varphi(\tau_+, -\gamma_+)} = \dfrac{a}{b}e^{2\gamma_+\tau_+}$ and $\lim\limits_{a\to 0}(\pi_+)'(a) = 0$, because $\lim\limits_{a\to 0} b = \lim\limits_{a\to 0} \pi_+(a) = b^* \neq 0$. Therefore substatement (a.1) is proved.

Now we observe that

$$(\pi_+)''(a) = \frac{d}{d\tau_+}\left(\frac{db}{da}\right)\frac{1}{\frac{da}{d\tau_+}} = -\frac{2d_+(1+\gamma_+^2)\sin^3\tau_+}{\beta_+\varphi(\tau_+, -\gamma_+)^3}(\sinh(\gamma_+\tau_+) - \gamma_+\sin\tau_+) > 0.$$

Therefore substatement (a.2) follows.

From expression (7) it follows that

$$\lim_{a\to\infty}\frac{\pi_+(a)}{a} = \lim_{\tau_+\to\pi}\frac{b(\tau_+)}{a(\tau_+)} = \lim_{\tau_+\to\pi} e^{2\gamma_+\tau_+}\frac{\varphi(\tau_+, -\gamma_+)}{\varphi(\tau_+, \gamma_+)} = e^{\gamma_+\pi}.$$

On the other hand by applying the L'Hôptal's rule it is easy to check that $\lim\limits_{a\to+\infty}(\pi_+(a) - e^{\gamma_+\pi}a) = t(1 + e^{\gamma_+\pi})/d$, which implies that the straight line $b = e^{\gamma_+\pi}a + t(1 + e^{\gamma_+\pi})/d$ is an asymptote of the graph of $\pi_+(a)$, we obtain substatement (a.3).

□

Proposition 6. *Consider the vector field X_+ in R_+ with a real focus equilibrium and such that $t_+ < 0$. Let π_+ be the map associated to the Poincaré map $\Pi_+ : L_+ \to L_+$ defined by the flow of the linear system $\dot{\mathbf{x}} = A_+\mathbf{x} + B_+$.*

(a) *Then the maps* π_+ *satisfy that* $\pi_+ : [a^*, \infty) \to [0, \infty)$, $\pi_+(a^*) = 0$, $\lim_{a\to\infty} \pi_+(a) = +\infty$ *and* $\pi_+(a) < a$ *in* (a^*, ∞).

(a.1) *If* $a \in (a^*, \infty)$ *then* $(\pi_+)'(a) = \dfrac{a}{\pi_+(a)} e^{2\gamma_+\tau_+}$. *Moreover* $(\pi_+)'(a) > 0$ *and* $\lim_{a\to a^*} (\pi_+)'(a) = +\infty$.

(a.2) *If* $a \in (a^*, \infty)$ *then* $(\pi_+)''(a) < 0$.

(a.3) *The straight line* $b = e^{\gamma_+\pi} a + t_+(1 + e^{\gamma_+\pi})/d_+$ *in the plane* (a, b) *is an asymptote of the graph of* π_+ *when* a *tends to* $+\infty$ *where* $\gamma_+ = t_+/\sqrt{4d_+ - t_+^2}$. *So* $\lim_{a\to\infty} (\pi_+)'(a) = e^{\gamma_+\pi}$.

Proof. The proof follows in a similar way to the proof of Proposition 5. □

Proposition 7. *Consider the vector field* X_o *in* R_o *with a virtual focus equilibrium. Let* π_o *be the map associated to the Poincaré map* $\Pi_o : D_o^* \subset L_- \to L_+$ *defined by the flow of the linear system* $\dot{\mathbf{x}} = A_o\mathbf{x} + B_o$ *from the straight line* L_- *to the straight line* L_+.

(a) *Then the map* π_o *satisfies that* $\pi_o : [0, \infty) \to [c^*, \infty)$, $c^* > 0$ *with* $\pi_o(0) = c^*$ *and* $\lim_{b\to\infty} \pi_o(b) = +\infty$.

(b) *If* $b \in [0, \infty)$ *then* $\pi_o'(b) = (1 - b_2)^2 \, e^{2\gamma_o\tau_o} \dfrac{b}{\pi_o(b)}$, *with* $\tau_o \to 0$ *when* $b \to \infty$ *and* $\lim_{b\to\infty} \pi_o'(b) = b_2 - 1$.

Proposition 8. *Consider the vector field* X_o *in* R_o *with a virtual focus equilibrium. Let* $\bar{\pi}_o$ *be the map associated to the Poincaré map* $\bar{\Pi}_o : \bar{D}_o^* \subset L_+ \to L_-$ *defined by the flow of the linear system* $\dot{\mathbf{x}} = A_o\mathbf{x} + B_o$ *from the straight line* L_+ *to the straight line* L_-.

(a) *Then the map* $\bar{\pi}_o$ *satisfies that* $\bar{\pi}_o : [d^*, \infty) \to [0, \infty)$, $d^* > 0$ *with* $\bar{\pi}_o(d^*) = 0$ *and* $\lim_{d\to\infty} \bar{\pi}_o(d) = +\infty$.

(b) *If* $d \in (d^*, \infty)$ *then* $\bar{\pi}_o'(d) = \bar{\pi}_o'(d) = (b_2 + 1)^2 \, e^{2\gamma_o\tau_o} \dfrac{d}{\bar{\pi}_o(d)}$, *with* $\tau_o \to 0$ *when* $d \to \infty$ *and* $\lim_{d\to\infty} \bar{\pi}_o'(b) = b_2 + 1$ *and* $\lim_{d\to d^*} \bar{\pi}_o'(d) = \infty$.

4 Limit cycles when X_o has a virtual focus in R_+

In what follows without loss of generality we will suppose that the focus of hypothesis (H2) is in R_+.

Proposition 9. *Assume that system* (4) *satisfies assumptions* (H1) *and* (H2) *and* $b_2 < -1$. *Then there exists a unique limit cycle of* (4), *which is hyperbolic. Moreover this limit cycle visits the three regions* R_-, R_o *and* R_+. *It is a repeller if* $t_o < 0$, *and an attractor if* $t_o > 0$.

Proof. We will give a sketch of proof.

Suppose that X_- has a virtual center and X_o has a virtual focus.

Using the previous notation we have $\gamma_i = \dfrac{\alpha_i}{\beta_i}$, $i \in \{-, o, +\}$. So $\gamma_- = 0$ and $\gamma_o, \gamma_+ \neq 0$. The domain of the first return map Π defined by

$$\Pi = \pi_o \circ \pi_- \circ \bar{\pi}_o \circ \pi_+$$

is an interval of $\mathbb{R}^+$ that depends on the domain of the mapping π_+ and $\bar{\pi}_o$ stated in Propositions 5, 6 and 8.

DOI: 10.5540/03.2015.003.01.0038

Suppose that $\gamma_o > 0$ and $\gamma_+ < 0$. In this case $\bar{D}_o(\bar{\pi}_o) = [d^*, \infty)$ where $d^* \geq 0$ and $\bar{\pi}_o(d^*) = 0$. This implies that $D(\Pi) = [k^*, \infty)$ where $k^* = \pi_+^{-1}(d^*)$. Moreover we have $\Pi : [k^*, \infty) \to [c^*, \infty)$ where $\Pi(k^*) = c^* = \pi_o(0)$.

Define the displacement function

$$h(a) = \Pi(a) - a.$$

Note that finding a fixed point of Π is equivalent to find zeroes of the function h.

The objective is to study the nature of the application Π and shown that this application is monotone in its domain. For this, we use that $\Pi' = \pi_o'(\pi_- \circ \bar{\pi}_o \circ \pi_+) \cdot \pi_-'(\bar{\pi}_o \circ \pi_+) \cdot \bar{\pi}_o'(\pi_+) \cdot \pi_+'$ and the expressions of each of its derivatives and signals given by the propositions 4, 5, 6, 7 and 8. Therefore, we conclude that if Π has a fixed point, it is unique.

$\square$

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DOI: 10.5540/03.2015.003.01.0038