Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 9, n. 1, 2022.

Trabalho apresentado no XLI CNMAC, Unicamp - Campinas - SP, 2022.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics
Preprint

On Bivariate Orthogonal Polynomials for Freud Weight Functions

Glalco S. Costa[1]
UFTM - ICTE, Uberaba, MG
Cleonice F. Bracciali[2]
UNESP - IBILCE, São José do Rio Preto, SP
Teresa E. Pérez[3]
Universidade de Granada, Granada, Espanha

Orthogonal polynomials with respect to the weight function in one variable $w(x) = |x|^\rho \exp(-|x|^m)$, $\rho > -1$, $m > 0$, was first studied by Géza Freud, [2]. A symmetric Freud weight function in one variable is usually given by

$$w_t(x) = e^{-x^4+tx^2}, \quad x \in \mathbb{R}, \tag{1}$$

where t is a real parameter. The corresponding moments exist, depend on t and are given by $\mu_k(t) = \int_{-\infty}^{+\infty} x^k e^{-x^4+tx^2} dx$, for $k = 0, 1, \ldots$.

Therefore, the sequence of orthonormal polynomials with respect to $w_t(x)$ are polynomials on the variable x whose coefficients depend on t, that we denote by $\{p_n(x,t)\}_{n\geqslant 0}$, they satisfy

$$\int_{-\infty}^{+\infty} p_n(x,t)p_m(x,t)w_t(x)dx = \delta_{n,m},$$

where $\delta_{n,m}$ are the Kronecker's delta. These polynomials satisfy the three term recurrence relation of the form

$$x\, p_n(x,t) = a_n(t)p_{n+1}(x,t) + a_{n-1}(t)p_{n-1}(x,t), \quad n \geqslant 0,$$

with $p_{-1}(x,t) = 0$ and $p_0(x,t) = \mu_0(t)^{-1/2}$.

The coefficients $a_n(t)$ satisfy the nonlinear difference equation

$$4\, a_n^2(t)[a_{n+1}^2(t) + a_n^2(t) + a_{n-1}^2(t)] - 2\, t\, a_n^2(t) = n + 1, \quad n \geqslant 0, \tag{2}$$

where $a_0^2(t) = \mu_2(t)/\mu_0(t)$ and $a_{-1}(t) = 0$ (see [1, 3]). The difference equation (2) coincides with the discrete Painlevé equation $\mathrm{dP_I}$

$$x_n(x_{n+1} + x_n + x_{n-1}) - \delta\, x_n = \alpha\, n + \beta + (-1)^n\, \gamma,$$

with $x_n = a_n^2(t), \alpha = \beta = 1/4, \gamma = 0, \delta = t/2$.

The connection between the coefficients of the three term recurrence relation for orthogonal polynomials in one variable and Painlevé equations is very well known (see for example, [4]).

[1]glalco.costa@uftm.edu.br
[2]cleonice.bracciali@unesp.br
[3]tperez@ugr.es

In this work we deal with the system of orthonormal polynomials in two variables $\{\mathbb{P}_n\}_{n\geq 0}$ given by

$$\mathbb{P}_n(x,y) = \begin{pmatrix} p_{n,0}(x,y) \\ p_{n-1,1}(x,y) \\ \vdots \\ p_{1,n-1}(x,y) \\ p_{0,n}(x,y) \end{pmatrix},$$

where $\{p_{m-k,k}(x,y)\}_{k=0}^{n}$ is a set of linear indepent polynomials of degree n.

The relation of orthonormality for $\{\mathbb{P}_n\}_{n\geq 0}$ is given by

$$< \mathbb{P}_n, \mathbb{P}_m^T >= (h_{ij})_{(n+1)\times(m+1)} = \begin{cases} I_{n+1,n+1}, & if \quad n = m, \\ [0]_{n+1\times m+1}, & if \quad n \neq m \end{cases}$$

where $[0]$ is the null matrix, for $i = 1, ..., n+1$ and $j = 1, ..., m+1$

$$h_{ij} = \int_{\mathbb{R}}\int_{\mathbb{R}} p_{n+1-i,i-1}(x,y)p_{m+1-j,j-1}(x,y)W(x,y)dxdy$$

and $W(x,y)$ is a weight function in two variables.

Here we consider the weight function

$$W(x,y) = e^{-q(x,y)}, \qquad (x,y) \in \mathbb{R}^2,$$

with $q(x,y) = a_{4,0}\,x^4 + a_{2,2}\,x^2\,y^2 + a_{0,4}\,y^4 + a_{2,0}\,x^2 + a_{0,2}\,y^2$ and $a_{i,j}$ are real parameters and the moments are defined by $\mu_{m,n} = \int_{\mathbb{R}}\int_{\mathbb{R}} x^n y^m W(x,y)dxdy$, for $n, m \in \mathbb{N}$.

The system of orthonormal polynomials $\{\mathbb{P}_n\}_{n\geq 0}$ satisfies the three term relations

$$x\mathbb{P}_n = A_{n,1}\mathbb{P}_{n+1} + A_{n-1,1}^T\mathbb{P}_{n-1},$$
$$y\mathbb{P}_n = A_{n,2}\mathbb{P}_{n+1} + A_{n-1,2}^T\mathbb{P}_{n-1},$$

for $n \in \mathbb{N}$, with $\mathbb{P}_{-1} = 0, \mathbb{P}_0 = \mu_{0,0}^{-1/2}$ and the coefficients $A_{n,i}$, $i = 1, 2$, are full ranked $(n+1)\times(n+2)$ matrices.

We analyse this system of bivariate orthonormal polynomials with respect to $W(x,y)$ and extend the difference equation (2) for the matrix coefficients of the three term relations of these polynomials, when $a_{2,0} = a_{0,2} = -t$,

Referências

[1] S. Belmehdi e A. Ronveaux. "Laguerre-Freud's equations for the recurrence coefficients of semi-classical orthogonal polynomials". Em: **Journal of Approximation Theory** 76 (1994), pp. 351–368.

[2] G. Freud. "On the coefficients in the recursion formulae of orthogonal polynomials". Em: **Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences** 76 (1976), pp. 1–6. DOI: 10.2307/20489026.

[3] A. Magnus. "On Freud's equations for exponential weights". Em: **Journal of Approximation Theory** 46 (1986), pp. 65–99. DOI: 10.1016/0021-9045(86)90088-2.

[4] W. Van Assche. **Orthogonal Polynomials and Painlevé Equations**. Cambridge University Press, 2018. ISBN: 978-1-108-44194-0.